Exactly Solvable Topological and Fracton Models as Gauge Theories

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1 Introduction

In this lecture, we are going to introduce some prototypical models with gapped topological order, symmetry protected topological order, and fracton order. In particular, we are going to introduce them from the perspective of gauging. First, we discuss a gapped topological model – the $2+1$D Toric code model, and show how it can be obtained by gauging the Z_2 global symmetry of a trivial paramagnet and is hence a Z_2 gauge theory. Next, we discuss a Symmetry Protected Topological model – a nontrivial paramagnet with global Z_2 symmetry that is fundamentally different from the trivial one – and show how gauging its Z_2 symmetry leads to a different Z_2 gauge theory with a different topological order. Finally, we are going to discuss a trivial paramagnet but with a different type of symmetry – rigid planar symmetries in $3+1$D – and how gauging this symmetry leads to a different type of order – the gapped fracton order.

2 The Toric Code Model

Let's first review the basics of the Toric Code model. Consider a square lattice with one spin $1/2$ degree of freedom (basis states $|0\rangle$ and $|1\rangle$) per each edge.

The Hamiltonian contains two types of terms: one involving four τ^x's around a vertex v, one involving four τ^z's around a plaquette p.

$$H = -\sum_v \left(\prod_{v\in e} \tau_e^x\right) - \sum_p \left(\prod_{e\in p} \tau_e^z\right) \tag{1}$$

τ^z and τ^x acts as

$$\tau^z|0\rangle = |0\rangle, \quad \tau^z|1\rangle = -|1\rangle, \quad \tau^x|0\rangle = |1\rangle, \quad \tau^x|1\rangle = |0\rangle \tag{2}$$

In the τ^x basis, $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$,

$$\tau^z|+\rangle = |-\rangle, \quad \tau^z|-\rangle = |+\rangle, \quad \tau^x|+\rangle = |+\rangle, \quad \tau^x|-\rangle = -|-\rangle \tag{3}$$

First we can check that all the terms in the Hamiltonian commute with each other. Second, as the Hamiltonian terms are all independent of each other, the ground state can be chosen to minimize

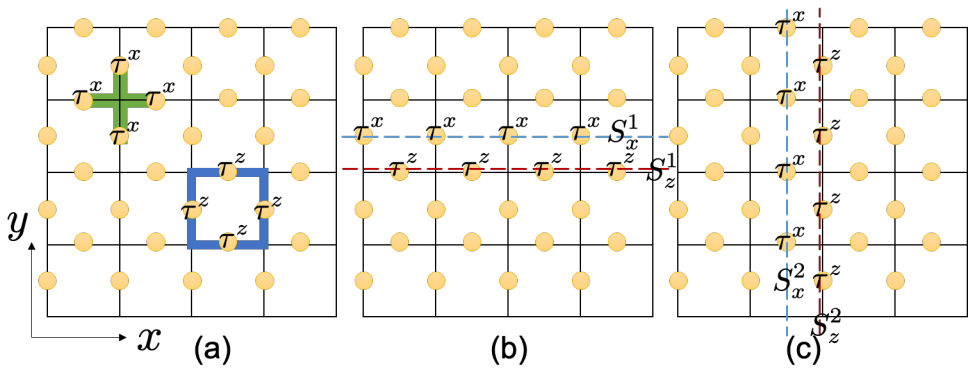


Figure 1: The Toric Code model: (a) Hamiltonian terms, (b) (c) logical operators along nontrivial loops.

energy of each of the Hamiltonian terms. There is a nice pictorial way to think of the wave-function. We can think of the spin 1/2 degrees of freedom as a Z_2 string on each edge. That is, the $|+\rangle$ state corresponds to no string on each edge while the $|-\rangle$ state corresponds to the existence of a string on the edge. To minimize the energy of the $-\prod_{v\in e}\tau_e^x$ terms, an even number of strings has to terminate at each vertex. That is, in the ground state, the strings always form loops and do not end.

Within the subspace of states representing closed loop configurations, the $\prod_{e\in p}\tau_e^z$ terms map one configuration to another by creating / destroying loops or moving them. The energy of the $-\prod_{e\in p}\tau_e^z$ terms can be minimized if the closed loop configurations that are related through the $\prod_{e\in p}\tau_e^z$ terms are added together into a superposition. Therefore, the ground state wave function takes the form

$$|\Phi\rangle = \sum_{\mathcal{C}:\text{ closed loop configurations}} |\mathcal{C}\rangle \tag{4}$$

Actually, in writing Eq. 4, we are a bit sloppy because there are situations where not all closed loop configurations can be mapped into each other through the moves generated by $\prod_{e\in p}\tau_e^z$. Consider, for example, the model defined on a torus. This is realized when we take the periodic boundary condition on the square lattice. The $2+1$D torus contains two nontrivial loops: one in the x direction, one in the y direction. The nontrivial loops satisfy the closed loop constraints (the $\prod_{v\in e}\tau_e^x$ terms) everywhere but a single one cannot be created with the $\prod_{e\in p}\tau_e^z$ terms. Therefore, there are four different ground states on torus: one with an even number of nontrivial loops in both x and y directions, one with an even number of nontrivial loops in x direction and odd in y, one with odd in x and even in y, one with odd in both x and y. The $S_z = \prod \tau^z$ string operator along the nontrivial loops in the x and y direction as shown in Fig. 1 (b) and (c) maps between these ground states. At the same time, the four ground states are eigenstates of the $S_x = \prod \tau^x$ string operators along the nontrivial loops on the dual lattice, as shown in Fig. 1 (b) and (c), with different eigenvalues $\{\pm 1, \pm 1\}$. Therefore, within the four dimensional ground state subspace, S_z^1, S_z^2, S_x^1, S_x^2 are the logical Pauli X and Pauli Z operators for the two logical qubits.

The S_z operator along any closed loop in the lattice commutes with the Hamiltonian. Same is true for the S_x operator along any closed loop on the dual lattice. When the string operator ends, Hamiltonian terms are violated and excitations are made in pairs (at each ends), as shown in Fig. 2. The ends of the S_z operator are the electric e excitations; the ends of the S_x operator are the magnetic m excitations. The most interesting feature of these excitations shows up when they

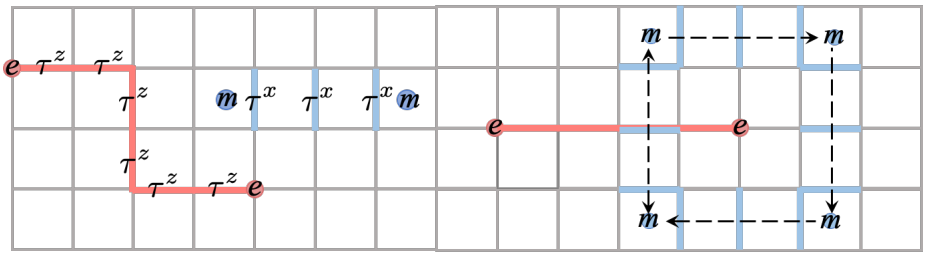


Figure 2: Creation of fractional excitations in the Toric Code model and their braiding.

go around each other. In particular, when a pair of m excitations are created and one of them is brought around a single e excitation and re-annihilated with the other m, as shown in Fig. 2, it results in a -1 phase factor.

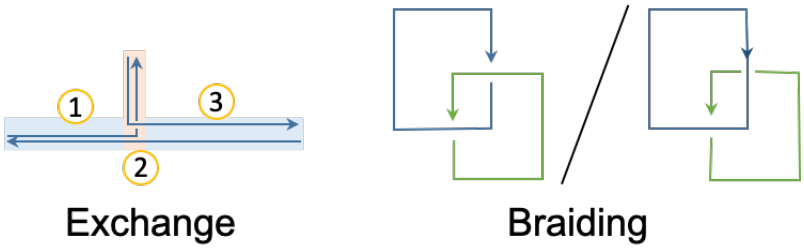


Figure 3: String operator configurations for calculating the topological spin and braiding statistics of fractional excitations.

More generally, the exchange statistics of one type of fractional excitation and the braiding statistics between two types of fractional excitations can be calculated using the diagrams in Fig. 3. The exchange statistics can be calculated in the three steps shown. The string operators in the same shaded areas overlap with each other. They are drawn as apart only for clarity. The braiding statistics is calculated as the ratio between two diagrams. The diagrams represent the same process as in Fig. 2 expect for the last (dispensable) step where the two e excitations are brought together and annihilated.

3 TC as a Gauge Theory

The Toric Code is also called a Z_2 gauge theory. Why is it a gauge theory? The first introduction to gauge theory in physics classes is probably through electromagnetism, which is a gauge theory because it has gauge symmetry. Recall that Maxwell's equations read

$$\nabla \cdot \vec{B} = 0,\ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t},\ \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0},\ \nabla \times \vec{B} = \mu_0 \vec{j} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \tag{5}$$

The gauge symmetry comes with the introduction of the electromagnetic potentials. Note that the first equation is automatically satisfied if

$$\vec{B} = \nabla \times \vec{A} \tag{6}$$

because $\nabla \cdot (\nabla \times \vec{A}) = 0$ for any $\vec{A}$. Substituting in the second equation we get

$$\nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0 \tag{7}$$

which can be automatically satisfied if

$$\vec{E} = -\nabla \varphi - \frac{\partial \vec{A}}{\partial t} \tag{8}$$

because $\nabla \times (\nabla \varphi) = 0$ for any φ.

The important observation here is that: given $\vec{E}$ and $\vec{B}$, $\vec{A}$ and φ are not uniquely determined. Instead, we get the same $\vec{E}$ and $\vec{B}$ if we change $\vec{A}$ and φ as

$$\vec{A} \to \vec{A} + \nabla f, \varphi \to \varphi - \frac{\partial f}{\partial t} \tag{9}$$

here f can be any real function of space and time. This is called a *gauge transformation* of the electromagnetic potentials φ and $\vec{A}$ and φ and $\vec{A}$ are called the *gauge fields*. Under the gauge transformation, the $\vec{E}$ and $\vec{B}$ fields remain invariant, hence the Maxwell's equation remains invariant. Therefore, the gauge transformation is a symmetry of the Maxwell's equation. Note that the gauge transformation is different from a global symmetry transformation in that the transformation can be different at different space time point. In this sense, it is called a *local symmetry* of E&M.

In the quantum version of the E&M theory, classical observables become quantum operators. In particular, to define the quantum E&M theory on a lattice, we put one rotor degree of freedom on each edge. A is the 'angular position' operator for the rotor DOF and takes value in $[0, 2\pi)$, while E is the 'angular momentum' operator for the rotor DOF and takes value in all integers $\mathbb{Z}$. E and A point along the direction of the edge and satisfy the commutation relation

$$[E, A] = -i \tag{10}$$

In putting the theory on a lattice, we break the space time equivalence of the theory. We can partially fix the gauge so that $\varphi = 0$ and focus only on the spatial part of the gauge transformation.

$$\vec{A} \to \vec{A} + \nabla f \tag{11}$$

In particular, if we choose f to be a Delta function $f(\mathbf{r}) = \mathbf{f_0} \delta_{\mathbf{r},\mathbf{r_0}}$ localized on a single lattice site $\mathbf{r_0}$, only the A's emanating from this site are affected by the transformation

$$A_{\langle \mathbf{r_0 r_1} \rangle} \to A_{\langle \mathbf{r_0 r_1} \rangle} - f_0 \tag{12}$$

where $\mathbf{r_1}$ are nearest neighbor sites to $\mathbf{r_0}$ and we have taken the edge between them to point away from $\mathbf{r_0}$. If the direction of the edge is reversed, the f_0 term gets a minus sign. Such a transformation can be induced by

$$U_{\mathbf{r_0}}(f_0) = \prod_{\mathbf{r_1}} e^{i f_0 E_{\langle \mathbf{r_0 r_1} \rangle}} \tag{13}$$

which is the gauge transformation centered on site $\mathbf{r_0}$. Note that A takes value in $[0, 2\pi)$, hence f_0 takes value in $[0, 2\pi)$, so this operator is well defined given that E takes integer value. Each local gauge transformation forms a group of $U(1)$. Therefore this is called a (quantum) $U(1)$ gauge theory. If we go back to the continuum, this operator becomes

$$e^{i f_0 \int_{S_{\mathbf{r_0}}} d\vec{S} \cdot \vec{E}} \tag{14}$$

where $S_{\mathbf{r_0}}$ is a small surface surrounding $\mathbf{r_0}$. Therefore, the system being invariant under the gauge transformations $e^{if_0 \int_{S_{\mathbf{r_0}}} d\vec{S}\cdot\vec{E}} = 1$ is equivalent to imposing Gauss's law at this point $\nabla \cdot \vec{E} = 0$.

Now let's consider the case where the $U(1)$ group is broken down to Z_2. The A operator now takes value of either 0 and π and the E operator takes value of either 0 and 1. It is more convenient to talk about the exponential of A and E. e^{iA} takes value in ± 1 and $e^{i\pi E}$ takes value in ± 1 as well. Their commutation relation becomes

$$e^{iA}e^{i\pi E} = -e^{i\pi E}e^{iA} \tag{15}$$

Therefore, we see that the gauge field DOF reduces to a qubit and e^{iA}, $e^{i\pi E}$ correspond to τ^z and τ^x respectively.

Now we can finally answer the question of why Toric Code is called a Z_2 gauge theory. Using the correspondence between e^{iA}, $e^{i\pi E}$ and τ^z and τ^x, we see that the Hamiltonian terms correspond to

$$\prod_{v\in e} \tau_e^x \to e^{i\pi \sum_{v\in e} E_e}, \quad \prod_{e\in p} \tau_e^z \to e^{i \sum_{e\in p} A_e} \tag{16}$$

Therefore, imposing these two terms correspond to imposing the gauge symmetry around vertex v and imposing the zero flux condition around plaquette p respectively.

Note that, in the case of classical E&M, gauge symmetry is enforced. That is, non-gauge invariant quantities are considered un-physical and are not allowed at all. While in the case of toric code (and the rotor model realization of quantum $E\&M$), gauge symmetry can be only imposed energetically. It will not be violated if the energy of the system is low enough. On the other hand, high energy states do exist in the total Hilbert space which violate the gauge symmetry. This is the price we have to pay in order to formulate gauge theory using local independent degrees of freedom.

4 TC from Gauging the Ising Paramagnet

As a Z_2 gauge theory, Toric code can be obtained by 'gauging' a model with global Z_2 symmetry. That is, to promote the global Z_2 symmetry in the model to a local Z_2 symmetry. In particular, it can be obtained by 'gauging' the trivial paramagnet in the transverse field Ising model. Let's see how that is achieved.

In the transverse field Ising model, the Hamiltonian at the paramagnetic limit (no spontaneous symmetry breaking) takes the simple form of

$$H = -\sum_v \sigma_v^x \tag{17}$$

where σ^x acts on the spin 1/2 degrees of freedom on each lattice site (blue dots in Fig.4) as $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This is the 'matter' DOF of the system. The system has a global Z_2 symmetry of $U = \prod_v \sigma_v^x$ and the ground state is invariant under this symmetry

$$|\psi\rangle = \otimes \frac{1}{\sqrt{2}} (|\uparrow\rangle + |\downarrow\rangle) \tag{18}$$

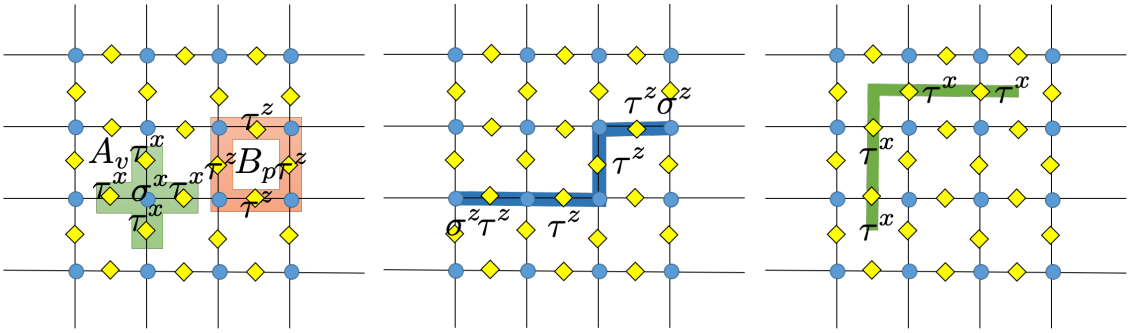


Figure 4: Gauging the Z_2 symmetry in the transverse field Ising model to obtain the Toric Code model.

where $|\uparrow\rangle$ and $|\downarrow\rangle$ are eigenstates of $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Our goal is to take the system with global symmetry and turn it into a system with local symmetry. That is, the symmetry group will be generated not by just one global symmetry operation, but by symmetry operations at all spatial locations (on each lattice site). Note that, the transverse field Hamiltonian as written in Eq. 17 is already invariant under local Z_2 symmetries generated by σ^x on each site. But this is no longer true if a small Ising perturbation of the form $\sigma^z\sigma^z$ is added. We want the procedure to work for the whole phase. Therefore, although we will focus on the transverse field limit for simplicity of discussion, we will present a gauging procedure that can be applied to any Z_2 symmetric Hamiltonian.

To do this, first we introduce the gauge field degrees of freedom on each edge of the lattice (yellow diamonds in Fig.4). For a Z_2 gauge field, the degrees of freedom are again two level spin 1/2's and we label them as τ spins. We define a local Z_2 symmetry generated at each lattice site with $U_v = \sigma_v^x \prod_{v\in e} \tau_e^x$ where the product is over all edges with v as one end point.

This local symmetry can be interpreted as enforcing the Gauss's law in the presence of charged matter. Using the correspondence between τ^x and $e^{i\pi E}$, we see that

$$U_v = \sigma_v^x \prod_{v\in e} \tau_e^x \to e^{i\pi\left(-n_v + \sum_{v\in e} E_e\right)} \tag{19}$$

where we have defined a Z_2 charge n_v for the σ spin as $e^{-i\pi n_v} = \sigma_v^x$. Therefore, imposing the local Z_2 symmetry is equivalent to imposing the Z_2 version of Gauss's law $\sum_{v\in l} E_l = n_v \mod 2$.

Next, we write down a new Hamiltonian which is invariant under the local symmetry transformations and also captures the dynamics of the matter and the gauge field. For the Ising paramagnetic model above, the Hamiltonian terms σ_v^x are already invariant under the local symmetries, so we do not need to do anything about them and can simply include them in the new Hamiltonian. Generally this is not the case. For example, if (small) nearest neighbor Ising coupling terms $\sigma_v^z\sigma_u^z$ are included in the Hamiltonian, they need to be modified as $\sigma_v^z\tau_{\langle vu\rangle}^z\sigma_u^z$ in order to be invariant under all local symmetry transformations. Such a modification is always possible for any local terms that satisfy the global Z_2 symmetry.

Besides that we add the term $U_v = \sigma_v^x \prod_{v\in e} \tau_e^x$ at every vertex v to enforce gauge symmetry (Gauss's law) and $F_p = \prod_{e\in p} \tau_e^z$, where the product is over all edges around a plaquette p, to enforce the

zero flux constraint on every plaquette. The total Hamiltonian then reads

$$H_g = -\sum_i \sigma_i^x - \sum_v U_v - \sum_p F_p = -\sum_i \sigma_i^x - \sum_v \sigma_v^x \prod_{v\in e} \tau_e^x - \sum_p \prod_{e\in p} \tau_e^z \tag{20}$$

If U_v is enforced as a constraint, or equivalently the weight of this term is taken to be $+\infty$, the Hamiltonian can be simplified into

$$H_g = -\sum_v \prod_{v\in e} \tau_e^x - \sum_p \prod_{e\in p} \tau_e^z \tag{21}$$

which is exactly the toric code Hamiltonian.

From the discussion in section 2, we knew that there are two types of bosonic excitations e and m with a mutual -1 statistics in Toric Code. This is of course still true if U_v is not enforced as a hard constraint and the matter DOF are kept in the system. The string operators can take the form shown in Fig. 4 (b), (c). The one of the form $\sigma^z\tau^z\tau^z...\tau^z\sigma^z$ creates a gauge charge e at its two ends, in a way that does not violate the gauge constraints. The one of the form $\tau^x\tau^x...\tau^x$ creates a π gauge flux m at its two ends. Using the procedure in Fig. 3, we see that both the gauge charge and gauge flux are bosons and they have a -1 mutual braiding statistics.

This gauging procedure can be generalized to all kinds of global internal unitary symmetries of group G in the following steps:

1. Take a system with global unitary internal symmetry of group G with degrees of freedom on the sites of a lattice and with global symmetry acting as a tensor product of operators on each lattice site.
2. Introduce gauge field degrees of freedom onto the edges of the lattice and define gauge symmetry transformation as acting on each lattice site and the edges around it.
3. Modify the terms in the original Hamiltonian, by coupling the original degrees of freedom with the gauge field, such that each term is invariant under all local gauge symmetry transformations.
4. Add vertex terms to the Hamiltonian to enforce gauge symmetry and add plaquette terms to enforce the zero flux constraint.

Question: 1. What do we get if we gauge the Z_2 symmetry in the symmetry breaking phase of the $2+1$D transverse field Ising model? 2. What do we get if we gauge the Z_2 symmetry in the $1+1$D transverse field Ising model?

5 Double Semion from Gauging the $\mathbb{Z}_2$ SPT

Now consider a different model with Z_2 global symmetry defined for the σ spins. This model is most easily defined on a triangular lattice and the σ spins live at each lattice site. We also discuss the triangular version of the Ising model for comparison.

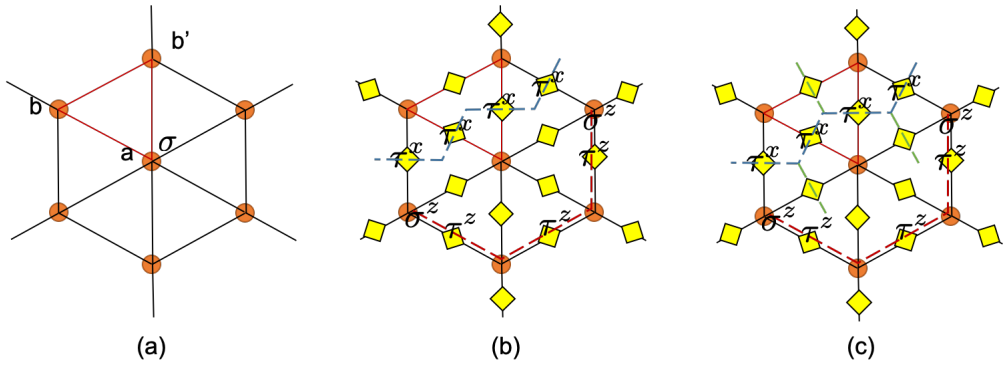


Figure 5: (a) Triangular lattice with one σ spin per site. (b) String operator for gauge charge and gauge flux excitations in H_g^0. (c) String operator for gauge charge and gauge flux excitations in H_g^1.

Consider a triangular lattice with one spin 1/2 per site, as shown in Fig. 5. We define two Hamiltonians. One is the transverse field limit of the Ising model H^0, the other H^1 is what we call a *Symmetry Protected Topological* (SPT) model with Z_2 symmetry. The meaning of the name will become clear later.

$$H^0 = -\sum_v \sigma_v^x \ , \quad H^1 = \sum_v \sigma_v^x \prod_{\langle vuu' \rangle} i^{\frac{1-\sigma_u^z \sigma_{u'}^z}{2}} \tag{22}$$

where the product in H^1 is over all triangles involving site v, as shown in Fig.5 (a). Both models are invariant under the Z_2 symmetry $\prod_v \sigma_v^x$. It can be checked that, for both models, different terms in the Hamiltonian commute with each other and the ground state can be found exactly. For H^0, the ground state is a product state of spins polarized in the x direction. Or put it another way, it is an equal weight superposition of all domain configurations with each domain pointing either in the $+z$ or $-z$ direction. On the other hand, it is probably not obvious, but the ground state of H^1 is again a superposition over all domain configurations but with phase factors. In particular, the configuration gets a -1 sign if there are an odd number of domain wall loops between domains.

$$|\psi^0\rangle = \sum_D |D\rangle \ , \quad |\psi^1\rangle = \sum_D (-1)^{N(D)} |D\rangle \tag{23}$$

where D labels domain configurations and $N(D)$ is the number of domain wall loops in this domain configuration. Note that, for both models, the ground state is unique and gapped and is symmetric under the Z_2 symmetry $\prod_v \sigma_v^x$.

Although these two models look different, at this point, we cannot tell whether they are fundamentally different or not – that is, whether they belong to different Z_2 symmetric phases. All the essential features we see so far – the gap and uniqueness of the ground state and its symmetry – do not rule out them being in the same phase. However, as we look closer, we are going to see that they actually belong to different phases when the global Z_2 symmetry is enforced. We are going to show this by applying the gauging procedure described above to both H^0 and H^1 and see how they give rise to different Z_2 gauge theories.

First we introduce Z_2 gauge field degrees of freedom τ on the edges. Gauge symmetry around each σ spin is defined as

$$U_v = \sigma_v^x \prod_{v \in e} \tau_e^x \tag{24}$$

The Hamiltonian terms in H^0 are already gauge symmetric. The terms in H^1 are not, but can be made gauge symmetric by replacing each $\sigma^z_u\sigma^z_{u'}$ by $\sigma^z_u\tau^z_{\langle uu'\rangle}\sigma^z_{u'}$. To obtain the fully gauged model, we add the flux term $F_p = \prod_{e\in p}\tau^z_e$ around each plaquette p in the original triangular lattice. And we add the gauge symmetry term as an energetic constraint. For H_0, we arrive at,

$$H^0_g = -\sum_v \sigma^x_v - \sum_v U_v - \sum_p F_p = -\sum_v \sigma^x_v - \sum_v \sigma^x_v \prod_{v\in e}\tau^x_e - \sum_p \prod_{e\in p}\tau^z_e \tag{25}$$

This is of almost the same form as the square lattice case we discussed above and we know the fractional excitations are the gauge charge excitation e created with a $\sigma^z\tau^z\tau^z...\tau^z\sigma^z$ string along the edges of the triangular lattice (the red dashed line in Fig. 5 (b)) and the gauge flux excitation m created with a $\tau^x\tau^x...\tau^x$ string along the edges of the dual honeycomb lattice (the blue dashed line in Fig. 5 (b)). e and m are both bosons, with a -1 mutual statistics between them.

For H_1, we get,

$$\begin{array}{rcl} H^1_g & = & \sum_v \sigma^x_v \prod_{<vuu'>} i^{\frac{1-\sigma^z_u\tau^z_{\langle uu'\rangle}\sigma^z_{u'}}{2}} \quad -\sum_v U_v \quad -\sum_p F_p \\ & = & \sum_v \sigma^x_v \prod_{<vuu'>} i^{\frac{1-\sigma^z_u\tau^z_{\langle uu'\rangle}\sigma^z_{u'}}{2}} \quad -\sum_v \sigma^x_v \prod_{v\in e}\tau^x_e \quad -\sum_p \prod_{e\in p}\tau^z_e \end{array} \tag{26}$$

The $\sigma^z\tau^z\tau^z...\tau^z\sigma^z$ type of string operator is still a legitimate string operator in the sense that it commutes with all the Hamiltonian terms along its length but violates some terms at its ends. Therefore, the excitation created by $\sigma^z\tau^z\tau^z...\tau^z\sigma^z$ – the gauge charge excitation e– is still a valid fractional excitation, which is a boson. On the other hand, the $\tau^x\tau^x...\tau^x$ type of string operator is no longer a legitimate string operator because it violates the $\sigma^x_v\prod_{<vuu'>} i^{\frac{1-\sigma^z_u\tau^z_{\langle uu'\rangle}\sigma^z_{u'}}{2}}$ terms along its length. This string operator needs to be modified to act not only on the edges of a path in the dual honeycomb lattice (the blue dashed line in Fig. 5 (c)), but also on the DOF besides it (the green dashed lines in Fig. 5 (c)). The exact form of the string operator is as follows:

$$S = \prod_{\langle vu\rangle\perp\beta}\tau^x_{\langle vu\rangle} \cdot \prod_{\langle vuu'\rangle,r} i^{\frac{1-\sigma^z_u\tau^z_{\langle uu'\rangle}\sigma^z_{u'}}{2}} \cdot \prod_{\langle vuu'\rangle,l}(-1)^{\tilde{s}_{vuu'}} \tag{27}$$

where β the dashed blue line. The first product runs over all edges $\langle vu\rangle$ crossing β. The next two products run over all triangles $\langle vuu'\rangle$ along the path such that u, u' are to the right of β or to the left of β respectively. $\tilde{s}_{vuu'} = \frac{1}{4}\left(1-\sigma^z_v\tau^z_{\langle vu\rangle}\sigma^z_u\right)\left(1+\sigma^z_v\tau^z_{\langle vu'\rangle}\sigma^z_{u'}\right)$.

This looks complicated, but we can learn something without worrying too much about all the details: the modification to the string operator comes in the form of some phase factors in the σ^z bases, which commutes with the gauge charge string operator. Therefore, the braiding statistics between the gauge charge and the gauge flux excitations remains -1. The modification phase factors play an important role in determining the self exchange statistics of the gauge fluxes. Careful computation shows that exchanging two gauge fluxes as in Fig. 3 results in a phase factor of i. The gauge fluxes are hence semions. The gauged model has the so-called 'Double Semion' topological order.

Therefore, H^0_g and H^1_g represent two different types of Z_2 gauge theories. They both contain gauge charge excitations which are bosonic and gauge flux excitations which braiding with the gauge charges with a -1 phase factor. However, the gauge flux excitations is a boson for H^0_g and a semion for H^1_g. They are two, and in fact the only two, distinct types of Z_2 gauge theory with bosonic gauge charge in $2+1$D. Correspondingly, H^0 and H^1 correspond to two different "symmetry protected topological" phases with Z_2 global symmetry. This is because any model with global Z_2 symmetry can be gauged. If the resulting gauge theories have different topological

order, then they cannot be smoothly connected as gauge theory. Correspondingly the models with global symmetry cannot be smoothly connected as long as the global symmetry is preserved, and hence are in different "symmetry protected" phases.

Some general conclusions that we can draw from the gauging procedure (applies to discrete symmetries):

1. The gauge charge comes from the symmetry charge. They are labeled by the representations of the group, are either a bosonic or a fermionic fractional excitation, depending on whether the symmetry charges in the original system are bosonic or fermionic.

2. The gauge flux comes from the symmetry flux and are labeled by the group elements of G (conjugacy classes if G is nonabelian). The statistics of the gauge flux is more complicated. It depends on the particular (symmetry protected) order of the original Hamiltonian. If the original Hamiltonian does not have nontrivial order (ground state can be a total product state), the gauge fluxes are bosonic.

3. The braiding statistics between a gauge charge and a gauge flux is independent of the original order. It is given by the Aharonov-Bohm phase factor.

One important feature of a symmetry protected topological model is its edge state. Consider a disc geometry as shown in Fig. 6. For H^0, the Hamiltonian can be naturally terminated on the boundary of the disc. As every Hamiltonian term is strictly on-site, we can simply keep all the terms inside the boundary and throw away the terms outside. The Hamiltonian terms on an open disc are still invariant under symmetry and commute with each other. Therefore, the ground state is unique, gapped and Z_2 symmetric. That is, the system is gapped and symmetric whether or not it has a boundary. In this case, we say that the edge state is gapped and symmetric.

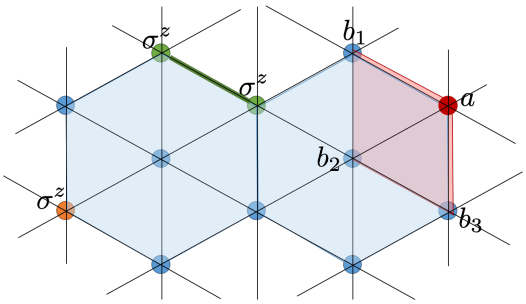


Figure 6: Z_2 symmetry protected topological model on an open disc.

For H^1, what people found was that, the edge state has to be gapless as long as symmetry is not broken either explicitly or spontaneously. Let me explain this statement more carefully. It is saying that if we put the system on a disc, as shown in Fig. 6, so that Hamiltonian terms inside the bulk (the two blue shaded hexagons) are the same as in H^1, but at or close to the boundary they might take a different form, the system (on the disc as a whole) will have a gapless spectrum, as long as each Hamiltonian term is symmetric and the ground state does not spontaneously break the symmetry. Moreover, the ground state differs from that on a surface without boundary only near the edge – that is, the reduced density matrices inside the bulk are the same but those near the edge will differ. Therefore, the gapless-ness comes from the "low energy modes" near the edge.

For H^1, it takes some effort to show this rigorously, which we will not do here. But we can get some hint by looking at what the Hamiltonian could look like with a boundary. Unlike H^0, each term in H^1 involves a number of spins. When we open a boundary, some of the terms fall completely within the boundary, some of the terms fall completely outside the boundary, while others stride across the boundary. We keep the terms completely within the boundary and throw away terms completely outside the boundary. For the terms that stride the boundary, if we throw them away, then the number of commuting Hamiltonian terms (with ± 1 eigenvalues) will be much smaller than the number of spins, resulting in a large degeneracy. The reduced density matrix in the bulk will be the same as in closed systems and the degeneracy comes entirely from the edge. Such degeneracy is not stable and can be easily split if some Hamiltonian terms are added back to the boundary.

For example, we can add σ_v^z terms (the orange dot in Fig. 6) to each of the spin on the boundary. This term commutes with terms in the bulk and commutes with each other. Therefore, it completely removes the degeneracy. Of course this is because it explicitly breaks the Z_2 symmetry.

Or we can add a $\sigma_v^z \sigma_u^z$ term (the green bond in Fig. 6) to every pair of nearest neighbor spins along the boundary. Such a term commutes with all the bulk terms and with each other and can remove the large degeneracy. But there is still a two fold degeneracy left because there is a non-trivial relation among all the boundary terms – their product is identity. The physical meaning of this two fold degeneracy is also clear. The $\sigma_v^z \sigma_u^z$ term is the Ising coupling between the Z_2 spins and leads to spontaneous symmetry breaking. So this is the case where the edge state is gapped but symmetry is (spontaneously) broken.

Another possibility is to add some variant of the original Hamiltonian term so that they fall within the boundary and are symmetric. For example, we can add a term around the red region in Fig. 6 centered around the red spin

$$\sigma_a^x \; i^{\frac{1-\sigma_{b_1}^z \sigma_{b_2}^z}{2}} \; i^{\frac{1-\sigma_{b_2}^z \sigma_{b_3}^z}{2}} \tag{28}$$

These terms can be written in a way that commute with all the bulk terms, so they act within the degenerate ground subspace of the bulk terms, but they no longer commute with each other. If more detailed calculation is carried out, we will see that the spectrum is gapless.

This is expected to be a generic feature of SPT phases: the edge state cannot be gapped and symmetric at the same time. This is of course what we see in, for example topological insulators, where the symmetry of interest are charge conservation symmetry and time reversal symmetry and the gapless mode is the helical edge mode.

6 The X-Cube Model from Gauging Subsystem Symmetries

Now let's turn to a different type of paramagnet and see how it gives rise to the 'fracton' order through gauging. We start from a $3+1$D spin $1/2$ model but with $2+1$D symmetries. That is, the symmetries are *subsystem* symmetries. Suppose that there is one spin $1/2$ degree of freedom on each lattice site of the cubic lattice. The Hamiltonian contains only a transverse field term

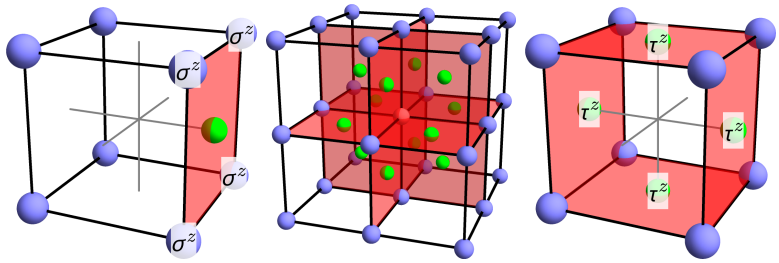
$$H = -\sum_v \sigma_v^x \tag{29}$$

and the ground state is gapped, unique and symmetric. There is one Z_2 symmetry generator on each xy, yz and zx plane.

$$U^{XY}_{z_0} = \prod_{v \in P^{XY}_{z_0}} \sigma^x_v \qquad U^{YZ}_{x_0} = \prod_{v \in P^{YZ}_{x_0}} \sigma^x_v \qquad U^{ZX}_{y_0} = \prod_{v \in P^{ZX}_{y_0}} \sigma^x_v. \tag{30}$$

$P^{XY}_{z_0}$ labels the xy plane with z direction coordinate z_0 and similarly for $P^{YZ}_{x_0}$ and $P^{ZX}_{y_0}$. Note that the manifolds that carry symmetry generators are rigid. This is what is called a 'higher-rank' symmetry, instead of a 'higher-form' symmetry where the manifolds that carry symmetry generators are non-rigid.

Although we are considering the same Hamiltonian as that of a paramagnet with global symmetry, the subsystem symmetry makes all the difference once some perturbation are allowed in the system. In the case of (for example $3+1$D) global symmetry, the minimal symmetric coupling allowed is a two-body term $\sigma^z_v \sigma^z_u$ which creates / annihilates a pair of Z_2 charges or hops one Z_2 charge from site v to site u. If instead we have $2+1$D subsystem symmetries in a $3+1$D system, a simple two-body Ising term is no longer symmetric under all the symmetries. Instead, the minimal symmetric coupling allowed are four-body terms $\sigma^z_v \sigma^z_{v+\hat{z}} \sigma^z_{v+\hat{y}} \sigma^z_{v+\hat{z}+\hat{y}}$ around a plaquette facing the x direction and similar terms around plaquettes facing x and y directions, as shown in Fig.7c (a). Therefore, with subsystem symmetry, Z_2 charges are created four at a time. Moreover, due to the lack of two-body Ising terms, once a charge is created, it cannot move by itself. This is the basic feature of a 'fracton', which we will see more clearly after gauging the subsystem symmetry in the model.



(a) (b) (c)

Figure 7: Gauging subsystem planar symmetry on a cubic lattice paramagnet.

The gauging procedure discussed in Section 4 cannot be directly applied because it applies to systems with global symmetry. To generalize the gauging procedure to systems with subsystem symmetry, we first need to decide what the gauge field look like and where to put them. Our goal is to promote the subsystem Z_2 symmetry into a local Z_2 symmetry, so naturally the gauge fields are Z_2 gauge fields, that is, qubit DOF with Pauli operators τ^x and τ^z.

To decide where to put the gauge fields, notice that in the case of global symmetry, the gauge fields are put on edges between two matter DOF to mediate the minimal coupling $\sigma^z_v \sigma^z_u$ along the edge $\langle vu \rangle$. Adding a gauge DOF τ on the edge allows us to modify the minimal coupling term to be $\sigma^z_v \tau^z_{\langle vu \rangle} \sigma^z_u$ so that it is invariant under gauge symmetry terms $\sigma^x_v \prod_u \tau^x_{\langle vu \rangle}$.

Now in $3+1$D with subsystem symmetry, the minimal Ising couplings are four-body plaquette terms as shown in Fig. 7 (a). Therefore, we want to add a gauge DOF τ to each plaquette and modify the minimal coupling terms to be $\sigma^z_v \sigma^z_{v+\hat{z}} \sigma^z_{v+\hat{y}} \sigma^z_{v+\hat{z}+\hat{y}} \tau^z_{p_{v,x}}$, where $p_{v,x}$ is the x direction facing plaquette surrounded by v, $v+\hat{z}$, $v+\hat{y}$, $v+\hat{z}+\hat{y}$, as shown in Fig. 7 (a). The gauge symmetry term is a product of one σ^x_v at vertex v and all the τ^x's that correspond to minimal coupling terms that contain vertex v. There are twelve such terms, hence the gauge symmetry is a product of thirteen

operators $U_v = \sigma_v^x \prod_{v\in p} \tau_p^x$, as shown in Fig. 7 (b). We can check explicitly that these two types of terms commute, that is, the modified minimal coupling terms are all gauge symmetric. Finally, let's add the 'flux' term which is a pure gauge 'vector potential' term made up of τ^z's so that it commutes with the gauge symmetry term. We can see that the simplest term that satisfies this property is a four-body term around the side plaquettes of a cube, as shown in Fig. 7 (c). There are three such terms around each cube F_c^{XY}, F_c^{YZ}, F_c^{ZX} and their product is identity.

The total gauged Hamiltonian is hence

$$H_g = -\sum_v \sigma_v^x - \sum_v U_v - \sum_c \left(F_c^{XY} + F_c^{YZ} + F_c^{ZX}\right) \tag{31}$$

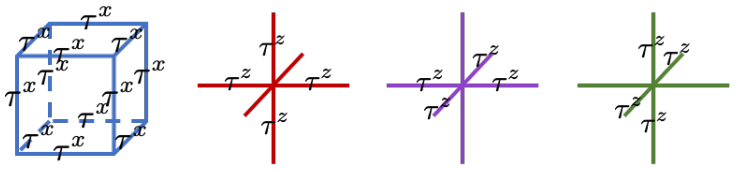


Figure 8: Hamiltonian terms in the X-cube model.

As discussed previously, we can enforce the U_v terms, and arrive at a model that involves only the gauge DOFs, which is the so-called "X-cube" model. To match the original form of the "X-cube" model, we take the dual lattice of the original cubic lattice so that the gauge DOFs are now on each edge. The model takes the form

$$H_{\text{X-cube}} = -\sum_c \prod_{e\in c} \tau_e^x - \sum_v \left(\tau^z_{v,v+\hat{x}}\tau^z_{v,v-\hat{x}}\tau^z_{v,v+\hat{y}}\tau^z_{v,v-\hat{y}} + \{\hat{y},\hat{z}\} + \{\hat{z},\hat{x}\}\right) \tag{32}$$

where the second and third term in the parenthesis can be obtained from the first term by replacing $\hat{x},\hat{y}$ by $\hat{y},\hat{z}$ and $\hat{z},\hat{x}$. The terms are shown in Fig. 8.

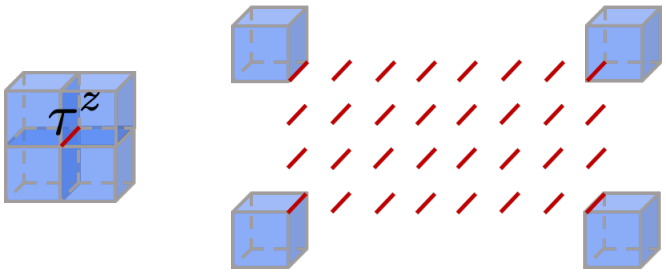


Figure 9: Creation and separation of gauge charges in the X-cube model.

Now let's see how fractional excitations are created. First, applying a single τ_e^z operator to one of the edges excites four cubic terms, as shown in Fig. 9. These are exactly the four charges involved in a minimal coupling term (note that vertices in the original lattice become cubes in the dual lattice). The cube excitations are hence the gauge charge excitations in the gauged model. Individual gauge charges cannot move. A set of four can separate as the corners of a rectangle.

The gauge flux excitation on the other hand is generated with τ^x operators. Applying a series of τ^x along the edges in x direction violates two of the three vertex terms at each end, as shown in Fig. 9. The color at the end points of the string operators indicate which vertex terms are violated. If a string of τ^x is applied in the y or z direction instead, a different set of vertex terms are violated.

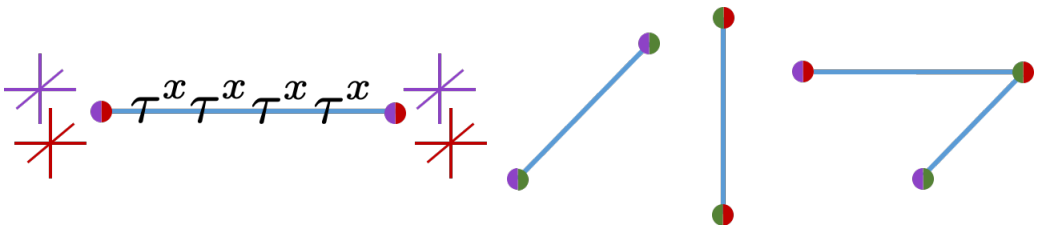


Figure 10: Creation and separation of gauge fluxes in the X-cube model.

Therefore, the gauge flux excitations are so-called 'lineons'. That is, they can move, but in one direction only. If they try to make a turn and move in a different direction, something will be left at the turning point.