Physics 129a **Homework 1** **Due 10/05/23 by 11:59pm**

0. Prove that every element in a group has a unique inverse. (Hint: the associativity condition of the composition rule may be helpful.)

1. Which of the following define groups? For those that do not explain the reasons. For those that do, find the identity and the inverse element. (You do not need to check all the group axioms.)

- The set of all complex $n \times n$ matrices when the group operation is matrix multiplication.
- The set of all complex $n \times n$ matrices, with a nonzero determinant, when the group operation is matrix multiplication.
- The set of all Hermitian $n \times n$ matrices M ($M^\dagger = M$), with a nonzero determinant, when the group operation is matrix multiplication.
- The set of all unitary $n \times n$ matrices ($M^\dagger M = I$) when the group operation is matrix multiplication.

2. Find a minimal generating set of the D_4 group, show that every element of the group can be generated from this generating set. "Minimal" means that if any element is taken out of the generating set, the set no longer generates the whole group.

3. A set of real-valued functions f_i of a real variable x can also define a group if we define multiplication as follows: given f_i and f_j, the product of $f_i \cdot f_j$ is defined as function $f_i(f_j(x))$. Show that the functions $I(x) = x$ and $A(x) = (1-x)^{-1}$ generate a three element group. Furthermore, including a function $C(x) = x^{-1}$ generates a six-element group.

4. Which of the following groups are isomorphic to each other. Give the explicit correspondence where an isomorphism exists:

- the complex numbers (1,i,-1,-i) with respect to multiplication;
- the integers (2,4,6,8) with respect to multiplication modulo 10;
- the permutations of four objects: no permutation (denoted as $()$), exchanging object 1 with 2 (denoted as $(1,2)$), exchanging object 3 with 4 (denoted as $(3,4)$), exchanging 1 with 2 and 3 with 4 (denoted as $(1,2)(3,4)$);
- the permutations of four objects: no permutation (denoted as $()$), cyclic permutation 1 to 2 to 3 to 4 (denoted as $(1,2,3,4)$), cyclic permutation 1 to 4 to 3 to 2 (denoted as $(1,4,3,2)$), exchanging 1 with 3 and 2 with 4 (denoted as $(1,3)(2,4)$).
- the four matrices $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ with respect to multiplication.