Physics 129a **Homework 4** **Due 10/26/23**

1. Show that the character of the direct product representation equals the product of the characters of the component representations. That is, if

$$D^{(\mu\times\nu)} = D^{(\mu)} \otimes D^{(\nu)} \tag{1}$$

then

$$\chi^{(\mu\times\nu)}(g) = \chi^{(\mu)}(g) \times \chi^{(\nu)}(g) \tag{2}$$

2. Recall all the properties we have derived for the quaternion group in homework 2.

a. If we are to write down the character table for the quaternion group, what is the size of the table? why?

b. What are the dimensions of the irreps?

c. Construct the character table for the quaternion group. You can use the following steps: 1. Write down the first row (corresponding to the trivial irrep) 2. For the rows corresponding to nontrivial $1D$ irreps, directly solve for the nontrivial $1D$ irreps and fill in the characters 3. use the orthogonality property to determine the character for the last row (corresponding to the $2D$ irrep).

Suppose that we take the direct product of two copies of the $2D$ irrep of the quaternion group.

d. what is the character of this direct product representation?

e. Decompose this direct product representation into irreps.

Answer these questions without using the explicit form of the $2D$ irrep, but only its character. (hint: use the result in problem 1.)

3. In this problem, we are going to prove the one to one correspondence between characters and equivalence classes of representations.

a. Suppose that D is a reducible representation of G. With certain invertible transformation S, we can put D into a block diagonal form. $SD(g)S^{-1} = \oplus_i D_i(g)$. Show that the character of D equals the sum of the characters of D_i.

b. Suppose that D and D' are two representations of the same dimension and they have the same character. Use the orthogonality property of characters to show that D and D' must decompose into the same set of irreps.

c. Show that D and D' are related by an invertible transformation $AD(g)A^{-1} = D'(g)$, hence are equivalent.

4. Consider the symmetry group of a regular triangle (D_3) as some linear transformation of the two dimensional vector space where the triangle is embedded. When the coordinates x and y undergo the linear transformation of the group, the quadratic form

$$a\frac{x^2}{\sqrt{2}} + bxy + c\frac{y^2}{\sqrt{2}}$$

is transformed into another quadratic form with different a', b', c'. (a', b', c') as a three dimensional vector is related to (a, b, c) by a 3×3 matrix. The 3×3 matrices obtained in this way form a representation of the D_3 group. Find the character of this representation. If the representation is reducible decompose it to its irreducible components. (You don't need to find the basis transformation to put the matrix into block diagonal form. Just state which irreps it should contain is enough.)