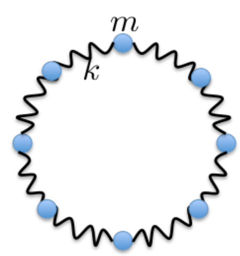
Physics 129a **Homework 5** **Due 11/2/23**

1. Coupled oscillator with translation symmetry.

Consider a system of N mass blocks coupled to each other with springs along a ring, as shown in the following figure. The mass blocks are constrained to move along the ring. The masses and springs are all the same. At equilibrium, all the springs are relaxed and they have the same length. The system has discrete translation symmetry.



Now consider small oscillation around the equilibrium configuration shown in the figure.

(1) How many degrees of freedom are there in the system? What are they?

(2) Write down the matrix T which represents elementary translation transformation on the degrees of freedom. (Hint: if the system is in equilibrium, the elementary translation maps one mass block to the next.)

(3) What symmetry group does translation form? List all the irreps of the symmetry group.

(4) What is the character of the representation found in (2)? When decomposed into irreducible blocks, what irreps does it contain?

(5) Find the eigenvector corresponding to each irrep. (Use the fact that the eigenvector transform under translation as the corresponding irrep.)

(6) If springs are added to connect second nearest-neighbor blocks while preserving translation symmetry, how do the eigenvectors change?

(7) Now take into consideration that the system is also symmetric under reflection. How do the eigenvectors transform under the reflection symmetry?

(8) What can we conclude about the eigen-frequencies when both reflection and translation symmetries are preserved?

2. $U(1)$ is the group of one dimensional unitary matrices (complex numbers of absolute value 1) and $SO(2)$ is the group of two dimensional orthogonal matrices with determinant 1. Their composition is matrix multiplication. Show that they are both isomorphic to the circle group.

3. z **direction angular momentum operator**

The angular momentum operator is defined as $\vec{L} = \vec{r} \times \vec{p}$. The z component of angular momentum is hence give by

$$L_z = xp_y - yp_x = -i\hbar \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) \tag{1}$$

Using the transformation from Cartesian coordinates (x, y, z) to spherical coordinates (r, θ, ϕ)

$$x = r\sin\theta\cos\phi,\ \ y = r\sin\theta\sin\phi,\ \ z = r\cos\theta \tag{2}$$

(1) Show that

$$L_z = -i\hbar\frac{\partial}{\partial\phi} \tag{3}$$

(2) Show that electron wave functions of the form $\Psi(r, \theta, \phi) = f(r, \theta)e^{im\phi}$ transforms as an irreducible representation under the $U(1)$ symmetry generated by the z direction angular momentum operator $R(\alpha) = e^{iL_z\alpha/\hbar}$.