Physics 129a **Homework 8** **Due 11/30/23**

1. The $SU(2)$ Group

In this problem, we will show that

$$\sigma_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

are the infinitesimal generators of the group of special unitary matrices of dimension two.

(1) For a two dimension complex matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, if it is unitary and has determinant 1, what conditions do a, b, c, d have to satisfy?

(2) Find a parameterization for all special unitary matrices of dimension two.

(3) Now take a linear combination of σ_x, σ_y and σ_z as $\sigma_{\vec{n}} = n_x\sigma_x + n_y\sigma_y + n_z\sigma_z$, where $\vec{n} = (n_x, n_y, n_z)$ is a real unit vector. Show that $(\sigma_{\vec{n}})^2 = \frac{1}{4}I_2$.

(4) Show that

$$R_{\vec{n}}(\theta) = e^{i\theta\sigma_{\vec{n}}} = \cos\left(\frac{\theta}{2}\right) + i2\sin\left(\frac{\theta}{2}\right)\sigma_{\vec{n}} \tag{2}$$

Hint: use the Taylor expansion of $e^{i\theta\sigma_{\vec{n}}}$.

(5) Show that the set of matrices you obtain in (2) coincide with the set of matrices you obtain in (4).

(6) Show that σ_x, σ_y and σ_z satisfy the same commutation relation as the infinitesimal generators J_x, J_y and J_z for $SO(3)$.

(7) Is the group $SU(2)$ isomorphic to $SO(3)$?

2. Spin 3/2 representation

Spin 3/2 forms a four dimensional irreducible representation of $SU(2)$. Use basis states

$$J_z|3/2\rangle = 3/2|3/2\rangle, J_z|1/2\rangle = 1/2|1/2\rangle, J_z|-1/2\rangle = -1/2|-1/2\rangle, J_z|-3/2\rangle = -3/2|-3/2\rangle, \tag{3}$$

Write down the angular momentum operator J_x, J_y, J_z as 4×4 matrices. (Use the formula in the lecture note for J_x and J_y.)

3. Addition of angular momentum

Consider the addition of a $j_1 = 1/2$ and a $j_2 = 1$ angular momentum.

(1) What angular momentum component is contained in this addition? That is, if we take the direct product of a $j_1 = 1/2$ irrep and a $j = 1$ irrep and decompose the composite representation into the direct sum of irreps, what irreps do we obtain? (Just apply the conclusion we obtained in class. You do not need to derive it from the character of the irreps.)

(2) Determine the Clebsch-Gordon coefficient for the addition of $j_1 = 1/2$ and $j_2 = 1$. Your answer may depend on some arbitrary choice of phase factors. Make the choice so that the the CG coefficients are all real. (There is still an ambiguity of $\pm$ signs. Either choice is ok.)

4. Electric dipole as vector operator

The electric dipole operator represents the potential energy of an electron in a uniform electric field $-e\vec{E}\cdot\vec{r} = -e(E_x x + E_y y + E_z z)$. While the dipole operator pointing in a particular direction breaks the $SO(3)$ rotation symmetry, the set of all dipole operators (of the same $|E|$) forms a representation of the $SO(3)$ rotation group. Rotation is generated by the angular momentum operator $\vec{J} = \vec{r} \times \vec{p}$, where $\vec{r} = (x, y, z)$ is the vector operator of spatial location (we also write it as (r_x, r_y, r_z)) and $\vec{p} = (p_x, p_y, p_z)$ is the vector operator of momentum.

(1) Use the commutation relation $[r_a, p_b] = i\delta_{ab}$ to show that $[J_a, r_b] = i\epsilon_{abc} r_c$.

This is similar to the commutation relation between angular momentum operators $[J_a, J_b] = i\epsilon_{abc} J_c$. Using this commutation relation between angular momentum operators, we showed in homework 7 problem 1 that $\vec{J}$ rotates as a three dimensional vector under rotation generated by $\vec{J}$. Convince yourself that the same conclusion applies to $\vec{r}$. That is, $\vec{r}$ rotates as a three dimensional vector under rotation generated by $\vec{J}$. You do not need to show work for this part.

(2) In fact we can interpret the commutation relation $[J_a, r_b] = i\epsilon_{abc} r_c$ as the action of J_a on r_b. Consider the three dimensional operator space with basis operators r_x, r_y and r_z. J_a can be interpreted as a linear transformation in this space. Write down the action of J_x, J_y, J_z as three dimensional matrices in this operator space.

(3) Which dipole operator remains invariant under rotation around z axis?

(4) Which dipole operator remains invariant under rotation around z axis up to a phase factor? (You may need to use complex E_x, E_y, E_z.)

(5) Which irrep of $SO(3)$ does the operator space form?