Introduction

introduction

What is a group? From Wikipedia: a group is a set equipped with an operation that combines any two elements to form a third element while being associative as well as having an identity element and inverse elements.

Example: A set of elements: integers n. If we add two integers, we get a third integer!

Let's check if the operation satisfies the four conditions called the group axioms, namely closure, associativity, identity and invertibility.

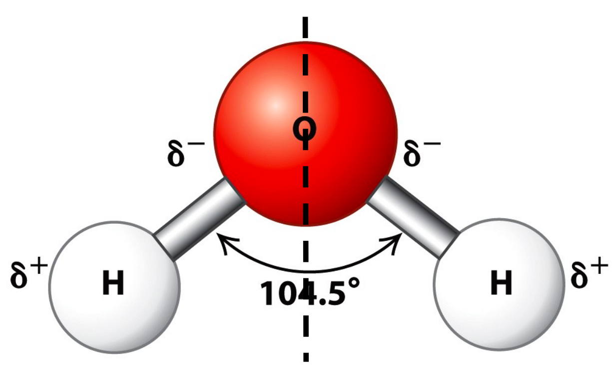
- Closure: Combining two elements in the set always produces a third element in the same set. $n + m = (n + m)$.
- Associativity: When combining more than two elements, the result does not depend on the order of combination. $(n + m) + p = n + (m + p)$.
- Identity: There exists an identity element e in the set such that combining this element with any element a in the set gives a. $0 + n = n + 0 = n$.
- Invertibility: For every element a in the set, there exist an element $-a$ such that the combination of the two gives e. $n + (-n) = (-n) + n = 0$.

Obviously, the set of integers, together with the addition operation, form a group, according to this definition.

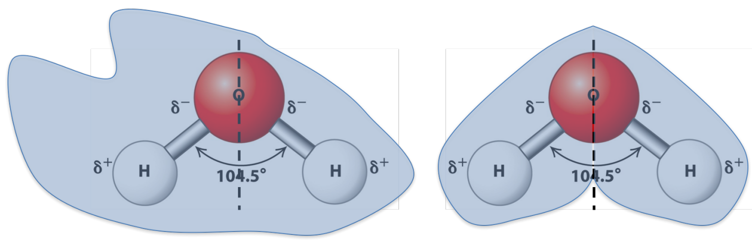
Why is group theory interesting to physics?

In physics, group theory is used to describe the symmetry of the physical system.

Example: reflection symmetry of the water molecule H2O.



If we want to find the distribution of the electron 'cloud' around the atoms, we need to solve the Schroedinger equation of an electron moving around one Oxygen and two Hydrogen atoms. That is a complicated task. However, we know that as the electron moves around the atoms in the water molecule, the potential energy it sees would be invariant under the left-right reflection operation (P). Without doing any of the calculation, intuition tells us the distribution is more likely to look like the figure on the right than the figure on the right because the figure on the right is more symmetric. That is, it is more compatible with the reflection symmetry of the atomic configuration. This is of course a very rough intuition. We will come back to this issue later in the class to specify exactly what we can learn about the electron cloud given the symmetry of the molecule. (We will learn for example, the configuration on the left is also possible, but the corresponding reflected configuration is equally possible.)



How is this related to group theory? It turns out the set of symmetry transformations, together with the combination operation of sequentially applying one transformation after another, form a group. In this particular example of water molecule, doing reflection P twice is equivalent to doing nothing (the identity operation I). Therefore, we have two elements in the group: the reflection operation P and the identity operation I. It is easy to check that the set formed by $\{I, P\}$ satisfies the above group axioms. Therefore, the set of symmetry operations $\{I, P\}$ form a group.

More generally, symmetry operations of a physical system always form a group and studying the group structure will help us to simplify the physical analysis at hand and understand better the dynamics of the system.

Actually, symmetry has become one of the guiding principles in formulating physical laws, because physics should be 'beautiful'. The following pictures contain examples of systems with different types of symmetry.

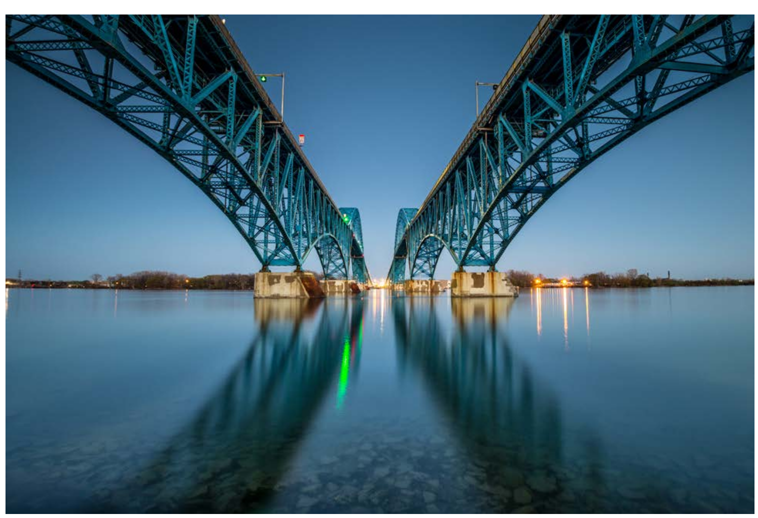


Figure 1: Y reflection symmetry.

Figure 2: X reflection symmetry.

Figure 3: Rotation symmetry.

Syllabus

This course aims to introduce the basic concepts of group theory and discuss the application of group theory in various fields of physics.

- General properties of groups (subgroup, coset, quotient group, homomorphism, direct product)
- Group representations (Irreducible representation, properties, CG coefficient)
- Finite groups (permutation groups, cyclic groups, classification)
- Lie groups ($SO(3)$, $SU(2)$, Lie algebra, general properties of simple Lie groups)
- Application: coupled harmonic oscillator

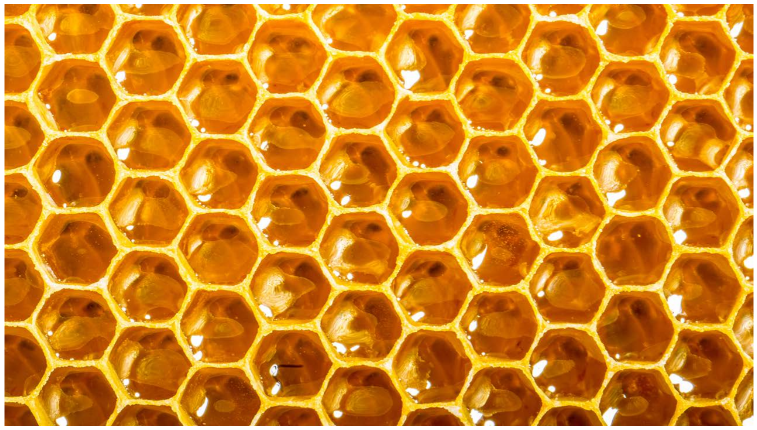


Figure 4: Translation symmetry.

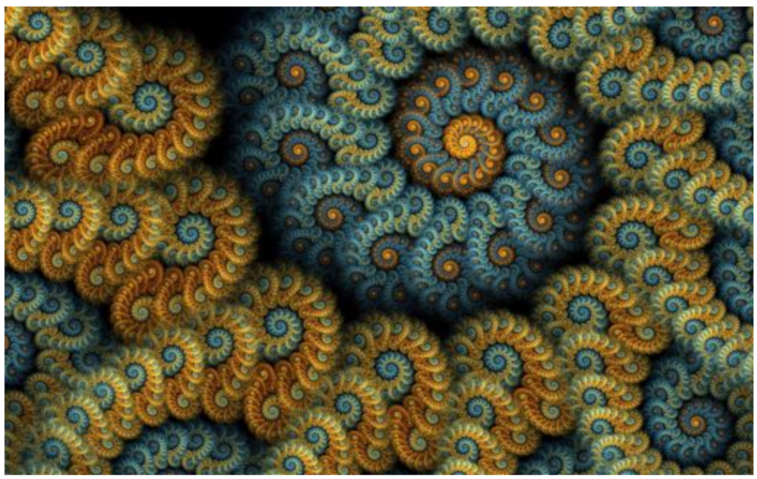


Figure 5: Scale invariance

- Application: atomic spectra
- Application: special relativity
- Application: condensed matter
- Application: standard model

Logistics

- Instructor: Xie Chen, Office: 163 W. Bridge, Phone: x3793, Email: xiechen@caltech.edu. Office hour: Tuesdays 4-5pm in Xie's office, with zoom option (starting Oct. 3).
- TA: Yixin Xu, yixinxu@caltech.edu; Abhishek Anand, abhi@caltech.edu. Recitation and Office hour: Wednesday, 4:30pm-6:30pm, Downs 107
- Prerequisites: Linear algebra, matrix; Differential equation; Basics of classical and quantum physics.
- Recommended textbooks:

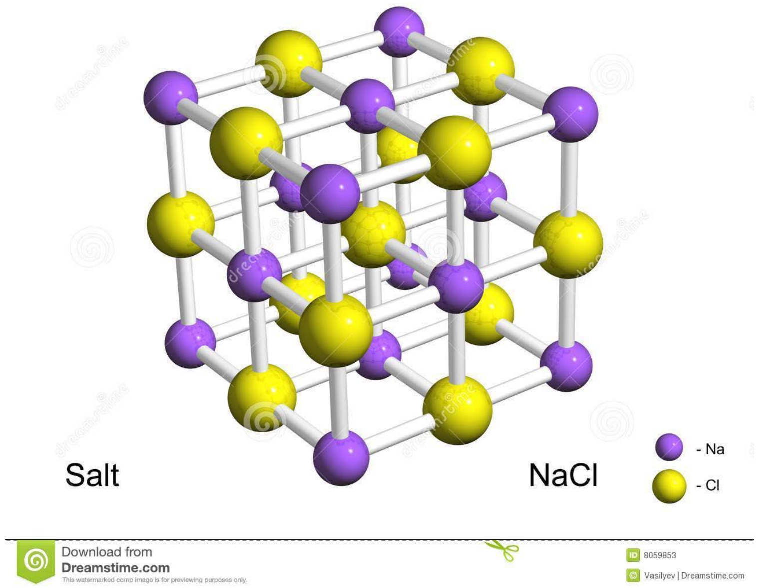


Figure 6: 3D lattice symmetry

not required, can be used for reference reading and homework.

- A. Zee, Groups, Group Theory in a Nutshell for Physicists, Princeton University Press, 2016.
- H. F. Jones, Groups, Representations, and Physics, Institute of Physics Publishing, 2nd ed., 1998.
- Wu-Ki Tung, Group Theory in Physics, World Scientific, 1985.
- J. F. Cornwell, Group Theory in Physics: An Introduction, Academic Press, 1997.
- E. P. Wigner, Group Theory, Academic Press, 1959.
- J. Talman, Special Functions, a Group-Theoretic Approach, Benjamin, 1968.
- H. Georgi, Lie Algebras in Particle Physics, Benjamin, 1982.
- P. Ramond, Group Theory: A Physicist's Survey, Cambridge Univ. Press, 2010. http://caltech.tind.io/record/740769?ln=en [Cambridge ebook]
- M. Tinkham, Group Theory and Quantum Mechanics, 1964.

- Problem Sets

  Posted on the website every Thursday, due on the following Thursday. Please scan or take a picture and upload on Canvas. Solutions will be posted also on the website.

- Exams

  Final exam in the last week of the term.

- Grading

  60% problem sets, 40% final exam.

- Extension policy

  One silver bullet for one week extension (no question asked). Please communicate with TA before using the silver bullet. For extra extensions, please email me.

1 Basic definition and properties of Groups

Definition: A group G is a set of elements $\{a, b, ...\}$ with a law of composition which assigns to each ordered pair $a, b \in G$ another element, written as ab, of G. The law of composition satisfies the following conditions:

1. Associativity: $\forall a,\ b,\ c \in G,\ a(bc) = (ab)c$.

2. Existence of identity: G contains an element, the identity element, denoted by e, such that $\forall a \in G,\ ae = ea = a$.

3. Existence of inverse: For all $a \in G$ there is an element, denoted by a^{-1}, such that $aa^{-1} = a^{-1}a = e$.

Comments:

1. In many cases, the composition is also called 'multiplication', but it does not have to be multiplication of numbers / matrices etc. It can be any operation that takes two elements and maps to a third one.

2. In general $ab \neq ba$, i.e. the operation of 'multiplication' is non-commutative. That is why the order of the elements of the pair a, b is important. If $ab = ba$ for all $a,\ b \in G$ the group is called Abelian. Otherwise, it is called Non-Abelian.

3. The definition includes the fact that the product ab is also a member of the set G. This is the closure axiom.

4. The number of elements in the group is called the order (cardinality) of the group.

Example: What is the identity element for each of the group below? Does every element has an inverse?

Group G	Composition Law
Z_n: integers modulo n	Addition (mod n)
Positive rational numbers	multiplication

Counter-Example: Why the following examples are not groups?

Set S	Composition Law
Z: integers	multiplication
R: real numbers (excluding ∞)	multiplication