4 Group representations

4.3 Examples

Example 1: C_2

The C_2 group has two possible normal subgroups: the identity group $\{e\}$ and C_2 itself. There are hence two possibilities for the kernel group and correspondingly two possibilities for the image group: C_2 and $\{e\}$.

When the image group is C_2, the representation is a faithful one. One possible realization is $D(e) = 1, D(c) = -1$.

When the image group is $\{e\}$, the representation is a trivial one, $D(e) = D(c) = 1$. Every group has a trivial representation like this.

There are also two dimensional representations of C_2. For example, $D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $D(c) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Example 2: D_3 represented as 2×2 real matrices.

Consider the two dimensional real vector space. The elements of D_3 can be taken to be linear transformations of the real vector space.

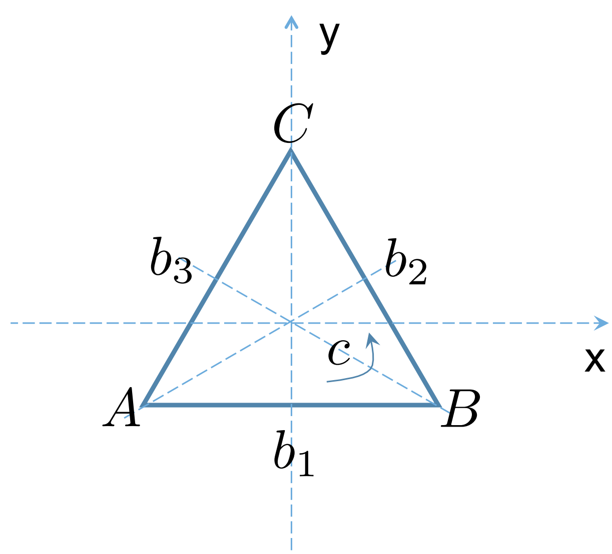

Figure 1: D_3 as linear transformation of 2D vector space

For example, c represents counter clockwise rotation by $2\pi/3$. A unit vector in the x direction

$(1,0)$ gets rotated to $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$. A unit vector in the y direction $(0,1)$ gets rotated to $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$. Therefore, the rotation can be represented by matrix

$$D(c) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \tag{1}$$

c^2 represents rotation by $4\pi/3$. The corresponding matrix is

$$D(c^2) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \tag{2}$$

It is straight forward to check that $D(c^2) = (D(c))^2$.

b_1 represents reflection across the y axis. The corresponding matrix is

$$D(b_1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3}$$

The matrix representing b_2 and b_3 are

$$D(b_2) = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \; D(b_3) = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \tag{4}$$

It is straight forward to check that this set of matrices give the same multiplication table as D_3. In fact, all the matrices are real and orthogonal. Therefore, this is an orthogonal representation of D_3.

Example 3: Circle group as rotation of 2D real vector space

The elements of the circle group can be taken to be continuous rotations of the $2D$ vector space. The group element labeled by θ corresponds to the 2×2 matrix

$$D(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{5}$$

This is again a real orthogonal representation.

Example 4: Circle group as rotation of 3D real vector space

We can also imagine that the circle group represent rotations of $3D$ real vector space around the z axis. The group element labeled by θ corresponds to 3×3 matrix

$$D(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6}$$

Example 5: Circle group as phase factors

With real matrices, the smallest faithful representation for the circle group is two dimensional. However, if we are allowed to use complex numbers, the circle group has a one dimensional faithful representation

$$D(\theta) = e^{i\theta} \tag{7}$$

4.4 Properties

1. Equivalent representations

Definition: Two n dimensional representations $D^{(1)}$ and $D^{(2)}$ of a group G are equivalent if *all* the matrices $D^{(1)}(g)$ and $D^{(2)}(g)$ are related by the same similarity transformation

$$D^{(1)}(g) = SD^{(2)}(g)S^{-1} \tag{8}$$

Comments:

(1) Equivalent representations have the same dimension.

(2) S is independent of g and this relation has to be satisfied for all g.

(3) This equivalence relation is consistent with the group property of the two representations. That is, if $D^{(2)}(g_1)D^{(2)}(g_2) = D^{(2)}(g_1g_2)$, we have

$$D^{(1)}(g_1)D^{(1)}(g_2) = SD^{(2)}(g_1)S^{-1}SD^{(2)}(g_2)S^{-1} = SD^{(2)}(g_1g_2)S^{-1} = D^{(1)}(g_1g_2) \tag{9}$$

(4) The relation between equivalent representations basically amounts to a basis change of the underlying (real or complex) vector space. Equivalent representations are considered the same.

(5) Maschke's Theorem: For finite (or more generally compact) groups, representations are always equivalent to unitary representations.

2. Character

If we want to have a way to characterize representations such that equivalent representations are characterized the same way, we can use the character.

Definition: The character of a representation D of a group G is the set $\chi = \{\chi(g)|g \in G\}$, where $\chi(g)$ is the trace of the representation matrix $D(g)$

$$\chi(g) = Tr(D(g)) = \sum_{i=1}^{n} D(g)_{i,i} \tag{10}$$

Comments:

(1) Matrices related by similarity transformation have the same trace

(2) Equivalent representations have the same character.

(3) Two representations with the same character are equivalent. This is a highly nontrivial and highly useful fact. We are not going to prove it now, but we are going to see how it comes up when we talk about irreducible representations.

(4) $\chi(e) = n$

(5) $\chi(g) = \chi(hgh^{-1})$, that is, conjugate elements have the same trace, each conjugacy class shares the same χ.

3. Reducibility

Example: circle group in three dimensional real vector space

Recall that the representation of the circle group in three dimensional real vector space reads

$$D(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{11}$$

All the $D(\theta)$ matrices take a block diagonal form. When they multiply, different blocks do not talk to each other. For the circle group, this is related to the fact that the z axis is invariant under the rotation operation, while vectors in the $x-y$ plane get rotated into each other. Therefore, this representation effectively 'decomposes' into two separate representations, a $2D$ representation $D^{(2)}$ acting on the $x-y$ coordinates and the $1D$ representation $D^{(1)}$ acting on the z coordinate, which are completely independent.

We write

$$D(\theta) = D^{(1)}(\theta) \oplus D^{(2)}(\theta) \tag{12}$$

$\oplus$ denotes direct sum, which is to combine matrices in block diagonal form.

Definition: A representation of dimension $n+m$ is said to be (completely) reducible or decomposible if there exists a basis transformation S such that $SD(g)S^{-1}$ is of the form

$$SD(g)S^{-1} = \begin{pmatrix} A(g) & O \\ O & B(g) \end{pmatrix} \tag{13}$$

for all $g \in G$, where A, B are sub-matrices of dimension $m \times m$ and $n \times n$ respectively. O is a null matrix (all entries 0) of dimension $n \times m$ and $m \times n$.

Comment: (1) $A(g)$ and $B(g)$ each form a representation of G. $A(g_1)A(g_2) = A(g_1g_2)$, $B(g_1)B(g_2) = B(g_1g_2)$.

(2) A and B could be decomposible themselves and we can continue the process until every block is irreducible.

(3) If we think of representation matrices as linear transformations of vector space, then blocks in a reducible representation correspond to closed subspaces under the linear transformations.

The irreducible representations provide building blocks for general decomposible representations and this is what we are going to focus on when we try to study in more detail about representations.

5 Irreducible Representations

Nick name: irreps.

5.1 Examples

(1) $C_2 = \{e, c\}$: the cyclic group of order two has two irreducible representations.

$$D(e) = 1, D(c) = 1 \text{ and } D(e) = 1, D(c) = -1 \tag{14}$$

Comments:

a. The first one is trivial (everything mapped to 1) while the second one is nontrivial.

b. Both of them are one dimensional.

c. If we consider the two irreps as two two-component vectors, they are orthogonal to each other.

(2) $D_3 = \{e, c, c^2, b_1, b_2, b_3\}$:

The dihedral group of order six has three irreps. The first one is trivial (everything mapped to 1). The two nontrivial ones are generated by:

$$D(c) = 1, D(b_1) = -1 \text{ and } D(c) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, D(b_1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{15}$$

Comments:

a. the D_3 group has two $1D$ irreps and one $2D$ irrep. In particular, the $2D$ representation cannot be decomposed into two $1D$ representations, because the $1D$ irreps acts trivially on the rotation subgroup while the $2D$ irrep is faithful.

b. We have specified the irreps just by the matrices of their generators. If we write out the matrices for all group elements, they are

irreps	e	c	c^2	b_1	b_2	b_3
$D^{(1)}$	1	1	1	1	1	1
$D^{(2)}$	1	1	1	-1	-1	-1
$D^{(3)}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

(16)