4 Group representations

5 Irreducible Representations

Nick name: irreps.

5.1 Examples

(1) $C_2 = \{e, c\}$: the cyclic group of order two has two irreducible representations.

$$D(e) = 1, D(c) = 1 \text{ and } D(e) = 1, D(c) = -1 \tag{1}$$

Comments:

a. The first one is trivial (everything mapped to 1) while the second one is nontrivial.

b. Both of them are one dimensional.

c. If we consider the two irreps as two two-component vectors, they are orthogonal to each other.

(2) $D_3 = \{e, c, c^2, b_1, b_2, b_3\}$:

The dihedral group of order six has three irreps. The first one is trivial (everything mapped to 1). The two nontrivial ones are generated by:

$$D(c) = 1, D(b_1) = -1 \text{ and } D(c) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, D(b_1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2}$$

Comments:

a. the D_3 group has two $1D$ irreps and one $2D$ irrep. In particular, the $2D$ representation cannot be decomposed into two $1D$ representations, because the $1D$ irreps act trivially on the rotation subgroup while the $2D$ representation is faithful.

b. We have specified the irreps just by the matrices of their generators. If we write out the matrices for all group elements, they are

irreps	e	c	c^2	b_1	b_2	b_3
$D^{(1)}$	1	1	1	1	1	1
$D^{(2)}$	1	1	1	-1	-1	-1
$D^{(3)}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

(3)

c. In $D^{(1)}$ and $D^{(2)}$, different group elements can be mapped to the same matrix (number). In $D^{(3)}$, different group elements are always mapped to different matrices. $D^{(3)}$ is said to be a faithful representation, while $D^{(1)}$ and $D^{(2)}$ are not faithful. The kernel of $D^{(1)}$ is the whole group, the kernel of $D^{(2)}$ is the rotation subgroup (a normal subgroup), and the kernel of $D^{(3)}$ is the identity element.

d. $D^{(1)}$ and $D^{(2)}$ are orthogonal six-component vectors. Is $D^{(3)}$ also orthogonal to them in some way?

The answer is yes. Let's take the element at position i, j $(i, j = 1, 2)$ from each of the matrices in $D^{(3)}$.

$$\begin{array}{ccccccc} & e & c & c^2 & b_1 & b_2 & b_3 \\ D^{(3)}[1,1] & 1 & -\frac{1}{2} & -\frac{1}{2} & -1 & \frac{1}{2} & \frac{1}{2} \\ D^{(3)}[1,2] & 0 & -\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ D^{(3)}[2,1] & 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ D^{(3)}[2,2] & 1 & -\frac{1}{2} & -\frac{1}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} \end{array} \tag{4}$$

The $D^{(3)}[i, j]$'s are pair-wise orthogonal. Moreover, they are orthogonal to $D^{(1)}$ and $D^{(2)}$ as well!

5.2 The fundamental orthogonality theorem

Suppose that $D^{(\mu)}$ and $D^{(\nu)}$ are two irreducible representations of a finite group G which are not equivalent if $\mu \neq \nu$ (but which are identical if $\mu = \nu$). Then

$$\frac{1}{|G|} \sum_{g \in G} \left(D^{(\mu)}(g)[j, k] \right)^* D^{(\nu)}(g)[s, t] = \frac{1}{d_\mu} \delta_{\mu\nu} \delta_{js} \delta_{kt} \tag{5}$$

This theorem is saying that the irreps of a group G are orthonormal in the following sense:

(1) For each irrep of G (in a fixed basis), take the $[i, j]$ entry in the representation matrix for all g and form a $|G|$-dimensional vector. ($|G|$ is the order of the group).

(2) If each irrep labeled by μ has dimension d_μ, then we have a set of $\sum_\mu d_\mu^2$ vectors.

(3) These vectors are all orthogonal to each other. That is, not only are vectors from different irreps orthogonal, vectors from the same irrep but different positions in the matrix are also orthogonal.

(4) The norm (length) of each vector is $|G|/d_\mu$.

(Check that this is the case for the irreps of D_3.)

This theorem puts a strong limit on the number and dimension of irreps. In particular

$$\sum_\mu d_\mu^2 \leq |G| \tag{6}$$

Actually, this inequality is saturated, as we will see later. (Check that this is the case for C_2 and D_3.)

5.3 Orthogonality of characters

Recall the definition and properties of character. The orthogonality relation for characters is obtained by taking suitable traces of the fundamental orthogonality theorem. Tracing over indices j, k and s, t, we get

$$\begin{aligned} & \tfrac{1}{|G|} \textstyle\sum_{g\in G} \sum_j \sum_s \left(D^{(\mu)}(g)[j,j]\right)^* D^{(\nu)}(g)[s,s] \\ = & \tfrac{1}{|G|} \textstyle\sum_{g\in G} \left(\sum_j D^{(\mu)}(g)[j,j]\right)^* \left(\sum_s D^{(\nu)}(g)[s,s]\right) \\ = & \tfrac{1}{|G|} \textstyle\sum_g \left(\chi^{(\mu)}(g)\right)^* \chi^{(\nu)}(g) \end{aligned} \tag{7}$$

On the other hand, we have

$$\begin{aligned} & \tfrac{1}{|G|} \textstyle\sum_{g\in G} \sum_j \sum_s \left(D^{(\mu)}(g)[j,j]\right)^* D^{(\nu)}(g)[s,s] \\ = & \textstyle\sum_j \sum_s \tfrac{1}{d_\mu} \delta_{\mu\nu}\delta_{js} = \delta_{\mu\nu} \end{aligned} \tag{8}$$

Therefore,

$$\frac{1}{|G|} \sum_g \left(\chi^{(\mu)}(g)\right)^* \chi^{(\nu)}(g) = \delta_{\mu\nu} \tag{9}$$

That is, the characters of different irreps are orthogonal to each other as $|G|$ dimensional vectors.

Comments:

(1) Although this theorem is derived from the previous one, it has the nice property that it does not depend on the particular basis chosen for the irreps.

(2) We can define an inner product between characters

$$\langle \chi^{(\mu)}, \chi^{(\nu)} \rangle = \frac{1}{|G|} \sum_g \left(\chi^{(\mu)}(g)\right)^* \chi^{(\nu)}(g) = \delta_{\mu\nu} \tag{10}$$

(3) A corollary of this theorem is that, the number of irreps must be smaller or equal to the number of conjugacy classes.

Proof: Recall that the character of group elements in the same conjugacy class is the same. Therefore, Eq. 9 can be re-written as

$$\frac{1}{|G|} \sum_i k_i \left(\chi_i^{(\mu)}\right)^* \chi_i^{(\nu)} = \delta_{\mu\nu} \tag{11}$$

where i labels conjugacy classes, k_i is the number of elements in the conjugacy class. Suppose that there are p conjugacy classes in all. Therefore, the set of vectors

$$\{\sqrt{k_1}\chi_1^{(\mu)}, \sqrt{k_2}\chi_2^{(\mu)}, ..., \sqrt{k_p}\chi_p^{(\mu)}\}, \forall \mu \tag{12}$$

form an orthogonal set in the p-dimensional vector space. Therefore, the number of different irreps is smaller or equal to the number of conjugacy classes. Actually, this inequality is again saturated (verify this for C_2 and D_3).

Using the orthogonality of the characters, we can decompose a general representation into its irreducible blocks.

Consider, for example, a $2D$ representation of the C_2 group

$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, D(c) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{13}$$

The character of this representation is $\chi = (2, 0)$. Decomposed into the characters of the irreps we get

$$\chi = (2, 0) = (1, 1) + (1, -1) = \chi^{(0)} + \chi^{(1)} \tag{14}$$

Therefore, there exist a basis transformation S such that

$$SD(g)S^{-1} = D^{(0)}(g) \oplus D^{(1)}(g) \tag{15}$$

We can check such a basis transformation is given by $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

5.4 Schur's Lemma

Let D and D' be two irreducible representations of a group G, of dimensions d and d' respectively, and suppose that there exists a $d \times d'$ matrix A such that

$$D(g)A = AD'(g), \forall g \in G. \tag{16}$$

Then only the following situations are possible:

(a) if $D(g)$ and $D'(g)$ are inequivalent representations, then $A = 0$;

(b) if $D(g)$ and $D'(g)$ are equivalent representations, then A is the similarity transformation between the two representations and $det(A) \neq 0$;

(c) In particular, if $D(g) = D(g')$ (are identical representations), then A must be proportional to the identity map.

Comments:

Schur's lemma applies only to irreps. It is not true for general reducible representations. For example, let's make a reducible representation of D_3 by taking a direct sum of $D^{(1)}$ and $D^{(2)}$ above.

$$D(g) = D^{(1)}(g) \oplus D^{(2)}(g),\ D(e) = D(c) = D(c^2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, D(b_1) = D(b_2) = D(b_3) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{17}$$

It is easy to see that $AD(g)A^{-1}$ can be true for all $g \in G$ with nonidentity matrix $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$.

While we are not going to present the proof for the lemma (which can be found in many textbook including Jones), we are going to discuss many of its important consequences.

All irreducible representations of an abelian group are one dimensional

Proof: Suppose that $D(g)$ is an irrep for abelian group G. Pick one particular element $g_0 \in G$. We have

$$D(g_0)D(g) = D(g)D(g_0), \forall g \in G \tag{18}$$

because G is abelian. Moreover, because we have assumed that $D(g)$ is irreducible, therefore, $D(g_0)$ must be proportional to the identity matrix. As this conclusion applies for any $g_0 \in G$, then for $D(g)$ to be irreducible, it has to be one dimensional. □