5 Irreducible representations

5.5 Regular representation and its decomposition into irreps

To see that the inequality

$$\sum_\mu d_\mu^2 \leq |G| \tag{1}$$

is saturated, we need to consider the so-called regular representation.

The regular representation is the representation of a group on 'itself'. Take all the group elements $g \in G$ and use them to label the orthonormal basis of a $|G|$ dimensional complex vector space. ($|G|$ is the order of the group.) Let's use the quantum mechanical notation and write the basis vectors as states $|g\rangle$. Notice that, now as basis state, the $|g\rangle$ can be put into linear combinations:

$$a_1|g_1\rangle + a_2|g_2\rangle + ... \tag{2}$$

This $|G|$ dimensional vector space is where the regular representation will act on.

Now we specify the linear transformation corresponding to each group element $g \in G$ on this vector space. As they are linear transformations, we only need to specify their action on the basis states:

$$D(g)|g_i\rangle = |gg_i\rangle \tag{3}$$

That is, the linear transformation left multiplies the label of the basis state and permutes them.

And we can check that the $D(g)$'s form a representation of the group G because

$$D(g_1)D(g_2)|g_i\rangle = |g_1g_2g_i\rangle = D(g_1g_2)|g_i\rangle \tag{4}$$

As this relation holds for every basis state $|g_i\rangle$, we have $D(g_1)D(g_2) = D(g_1g_2)$.

Example: the regular representation of C_2 is

$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D(c) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5}$$

Q: is this irreducible? If so, why? If not, what are the irreducible blocks?

The regular representation is a very special representation. In the $|g_i\rangle$ basis, the representation matrices contains only 0's and 1's and each row and each column contains only one 1. In $D(e)$, all the 1's are on the diagonal. In $D(g), g \neq e$, all the 1's are off diagonal. Therefore we have the following conclusion

$$\chi(e) = |G|, \chi(g) = 0 \tag{6}$$

The regular representation is reducible (how to tell?) In fact, it contains every irreducible representation of the group and each irrep μ appears d_μ times where d_μ is the dimension of the irrep μ. Let's try to see why this is the case.

Let's suppose that we can decompose $D(g)$ into irreps $D^{(\mu)}$

$$D(g) = \oplus_\mu \, a_\mu D^{(\mu)}(g) \tag{7}$$

a_μ is the integer representing the number of times each irrep appears.

How to find a_μ? Let's take the trace on both sides

$$\chi(g) = \sum_\mu a_\mu \chi^{(\mu)}(g) \tag{8}$$

where we have used the fact that the trace of the direct sum of matrices equals the sum of their trace.

In order to find out each a_μ from this sum, we take an inner product between the character of the irrep μ and the character of the regular representation

$$\langle \chi^{(\nu)}, \chi \rangle = \sum_\mu a_\mu \langle \chi^{(\nu)}, \chi^{(\mu)} \rangle \tag{9}$$

As the $\chi^{(\mu)}$ are orthonormal, we have

$$\langle \chi^{(\nu)}, \chi \rangle = a_\nu \tag{10}$$

On the other hand, $\chi = (|G|, 0, 0, 0, ...)$, therefore its inner product with $\chi^{(\nu)}$ gives

$$\langle \chi^{(\nu)}, \chi \rangle = \frac{1}{|G|} |G| \chi^{(\nu)}(e) = d_\nu \tag{11}$$

Hence we have $a_\nu = d_\nu$. That is, the regular representation contains every irrep μ d_μ times.

Now finally let's look at Eq.8 again for the special case of $g = e$. We get

$$|G| = \chi(e) = \sum_\mu a_\mu \chi^{(\mu)}(e) = \sum_\mu a_\mu d_\mu = \sum_\mu d_\mu^2 \tag{12}$$

That is, the inequality $\sum_\mu d_\mu^2 \leq |G|$ is saturated.

In fact, the inequality that the number of irreps is smaller or equal to the number of conjugacy classes is also saturated (we don't prove it here).

Number of Irreps = Number of Conjugacy Classes.

5.6 The character table

Now with many theorems at hand, one can try to find all the irreps of a group. This is a legitimate question, but it has infinitely many different answers because the representation matrices depend on the choice of basis. On the other hand, the character of irreps are independent of basis choice and have a one to one correspondence with the equivalence class of irreps. Therefore, it suffices to find all the orthonormal characters of the irreps of a group.

The characters of the irreps of a group are usually listed in a table where each row corresponds to an irrep and each column corresponds to a conjugacy class (recall that elements in the same conjugacy class have the same trace). Because the number of irreps equals the number of conjugacy classes, this is a square table. To find the entries in the table, we make use of the following facts:

(1) $\sum_\mu d_\mu^2 = |G|$. Because $\chi^\mu(e) = d_\mu$, the norm of the first column of the table (corresponding to e) is $|G|$.

(2) $\langle \chi^{(\mu)}, \chi^{(\nu)} \rangle = \frac{1}{|G|} \sum_g \left(\chi^{(\mu)}(g)\right)^* \chi^{(\nu)}(g) = \delta_{\mu\nu}$.

As in the character table the character of elements in the same conjugacy class only appears once, we have

(3) $\sum_i k_i (\chi_i^{(\mu)})^* \chi_i^{(\nu)} = \delta_{\mu\nu}|G|$. Here i labels different conjugacy classes and k_i denotes the number of elements in a conjugacy class.

Now let's see some examples:

Character table of C_3:

The C_3 group is an abelian group of order three. The three elements are $\{e, c, c^2\}$ and each form a conjugacy class. Therefore, there are three irreps and each of them being one dimensional. The character table would be a 3×3 matrix.

The first irrep is the trivial one, with every element in the character table being 1. The second and third irreps are nontrivial. The number corresponding to c needs to have order 3. Therefore, it can only be $\omega = e^{i2\pi/3}$ and $\bar{\omega} = e^{-i2\pi/3}$. The two possibilities correspond to the other two irreps.

C_3	e	c	c^2
$\chi^{(1)}$	1	1	1
$\chi^{(2)}$	1	ω	$\bar{\omega}$
$\chi^{(3)}$	1	$\bar{\omega}$	ω

(13)

Comments:

(1) $D^{(1)}$ is a trivial representation while $D^{(2)}$ and $D^{(3)}$ are faithful representations.

(2) Their orthogonality can be checked in a straight forward way.

Once we have the character table, we can determine if any given representation is reducible and if so what are the irreducible blocks. For example, consider the $2D$ representation of C_3 as rotations of a two dimensional plane. The representation matrices are

$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; D(c) = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, \; D(c^2) = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \tag{14}$$

Taking the trace of these matrices we get

$$\chi(e) = 2, \chi(c) = -1, \chi(c^2) = -1 \tag{15}$$

As a three dimensional vector, it can be decomposed as

$$\chi = \chi^{(2)} + \chi^{(3)} \tag{16}$$

Therefore, this $2D$ representation is reducible and contains two irrep blocks equivalent to $D^{(2)}$ and $D^{(3)}$.

Character table of D_3:

Now let's discuss a more interesting example, the D_3 group. The D_3 group contains six group elements $\{e, c, c^2, b_1, b_2, b_3\}$ which belong to three conjugacy classes $\{e\}$, $\{c, c^2\}$, $\{b_1, b_2, b_3\}$. Therefore, the character table is also 3×3.

The first representation is trivial with every element mapped to 1. Therefore, the characters in the first row are all 1.

Using the relation that $\sum_\mu d_\mu^2 = |G|$, we know that there are two 1D irreps and one 2D irrep in total.

Next we can try to solve for the other $1D$ representation. As c is an order 3 element, it can only be mapped to $1, \omega, \bar{\omega}$. On the other hand, as the b_i's are order 2 elements, they can only be mapped to ± 1. Because $cb_1 = b_2$, it is not possible for c to be ω or $\bar{\omega}$. It has to be 1. Then the b_i' s are either all 1 which corresponds to the trivial irrep or all -1 which corresponds to another $1D$ irrep. The character of the second irrep is $\{1, 1, -1\}$. We can check that the first two irreps are orthogonal (remember to count the multiplicity of the conjugacy classes).

We know a $2D$ representation which is obtained by thinking of D_3 as linear transformations of the two dimensional plane. This representation is generated by

$$D^{(3)}(c) = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, \; D^{(3)}(b_1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{17}$$

from which we find the corresponding character to be $\chi^{(3)}(e) = 2, \chi^{(3)}(c) = -1, \chi^{(3)}(b) = 0$. This is orthogonal to the previous two characters. Therefore we know that this must be the third irrep we are looking for.

If we do not know of the explicit form of the $2D$ irrep, we can still solve for its character. Because it is two dimensional $\chi(e) = 2$. Suppose that $\chi(c) = \alpha$ and $\chi(b) = \beta$. Due to the orthogonality condition, we have

$$2 + 2\alpha + 3\beta = 0, 2 + 2\alpha - 3\beta = 0 \tag{18}$$

from which we get $\alpha = -1$ and $\beta = 0$. The norm of the character is 6 as it should be.

D_3	$\{e\}$	$\{c, c^2\}$	$\{b_1, b_2, b_3\}$
$\chi^{(1)}$	1	1	1
$\chi^{(2)}$	1	1	-1
$\chi^{(3)}$	2	-1	0

(19)

5.7 Direct sum of representations

Starting from two representations $D^{(1)}(g)$ and $D^{(2)}(g)$ of dimensions m and n, we can form a bigger representation of dimension $m+n$ by taking a direct sum of them.

$$D^{(1+2)}(g) = D^{(1)}(g) \oplus D^{(2)}(g) = \begin{pmatrix} D^{(1)}(g) & \\ & D^{(2)}(g) \end{pmatrix} \tag{20}$$

If $D^{(1)}(g)$ and $D^{(2)}(g)$ are representations, $D^{(1+2)}(g)$ is also a representation and their characters satisfy

$$\chi^{(1+2)}(g) = \chi^{(1)}(g) + \chi^{(2)}(g) \tag{21}$$

5.8 Direct product of representations

Moreover, we can multiply two representations and get a new representation, as defined below.

Suppose that we have two representations $D^{(\mu)}(g)$ and $D^{(\nu)}(g)$ of dimensions m and n respectively, the direct product of the two representations is a $m \times n$ dimensional representations and its matrix elements are given by

$$D^{(\mu\times\nu)}(g)_{ab,cd} = D^{(\mu)}(g)_{a,c} \times D^{(\nu)}(g)_{b,d} \tag{22}$$

Here ab and cd are two-digit numbers where the first digit takes m different values while the second digit takes n different values. This is denoted as $D^{(\mu\times\nu)} = D^{(\mu)} \otimes D^{(\nu)}$. Can you show that $D^{(\mu\times\nu)}(g)$ forms a representation if $D^{(\mu)}(g)$ and $D^{(\nu)}$ both form representations?

The character of the direct product representation satisfies

$$\chi^{(\mu\times\nu)}(g) = \chi^{\mu}(g) \times \chi^{\nu}(g) \tag{23}$$

We are going to prove this in the homework.

Example: The cyclic group C_n.

The irreps of the cyclic group are all one-dimensional, given by $D^{(k)} = (1, e^{i2\pi k/n}, e^{i4\pi k/n}, ..., e^{i2\pi(n-1)/n})$, for $k = 0, 1, ..., n-1$. If we take the direct product of $D^{(k)}$ with $D^{(k')}$, we get the irrep of $D^{(k+k')\text{mod } n}$. Therefore, the irreps of a cyclic group forms the same cyclic group. This is in general true for any abelian finite groups.

Example: Consider the $2D$ irrep of D_3 with generators

$$D^{(3)}(c) = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}, \quad D^{(3)}(b_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{24}$$

The direct product of two copies of this $2D$ irrep is generated by

$$D^{(3\times3)}(c) = \frac{1}{4}\begin{pmatrix} 1 & -\sqrt{3} & -\sqrt{3} & 3 \\ \sqrt{3} & 1 & -3 & -\sqrt{3} \\ \sqrt{3} & -3 & 1 & -\sqrt{3} \\ 3 & \sqrt{3} & \sqrt{3} & 1 \end{pmatrix}, \quad D^{(3\times3)}(b_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{25}$$

It is easy to verify that this is indeed another representation of the group, but a reducible one. To see what irrep blocks it contains, we calculate the character of this representation as

$$\chi(e) = 4, \chi(c) = 1, \chi(b) = 0 \tag{26}$$

which, according to the character table of D_3, can be decomposed into

$$\chi = \chi^{(1)} + \chi^{(2)} + \chi^{(3)} \tag{27}$$

Therefore, this direct product representation can be decomposed into three different irreps, each appearing once.

Note that for D_3, or nonabelian groups in general, the direct product of irreps are in general not irreps, but can be decomposed into a direct sum of irreps. This is a structure beyond groups.

The physical meaning of the direct product representation is very different from that of the direct sum representation. The direct sum describes the symmetry transformation of a single object, but with states in different subspaces; the direct product describes the symmetry of a composite system, with more than one objects transforming under the symmetry. For example, consider the electrons orbiting a nucleus. The s orbital and p orbital both transform under the rotation symmetry of the system and each form a representation $D^{(s)}(g)$ and $D^{(p)}(g)$ of the group. If we want to describe the symmetry transformation of an electron which can live both in s orbital or p orbital (or some mixture of them), we use the direct sum $D^{(s)}(g) \oplus D^{(p)}(g)$; on the other hand, if we want to describe the symmetry transformation of two electrons in the s orbital, we use the direct product $D^{(s)}(g) \otimes D^{(s)}(g)$. (What if we want to describe two electrons that can live in both s and p orbitals?)