

## 5 Irreducible representations

### 5.9 Irreps of the circle group and charge

We have been talking mostly about finite groups. Continuous groups are different, but their representation theory can be similar in many ways. Let's consider the simplest case of a continuous group: the circle group.

The circle group is abelian, therefore all of its irreps are one dimensional. The trivial one maps every group element labeled by  $\theta \in [0, 2\pi)$  to 1.

A nontrivial representation is given by  $D^{(1)}(\theta) = e^{i\theta}$ . Actually, the circle group has infinitely many different irreps given by

$$D^{(n)}(\theta) = e^{in\theta}, n \in \mathbb{Z} \quad (1)$$

While the group elements in the circle group are continuous, its irreps are discrete, labeled by integers  $n$ , which is usually referred to as 'charge'. This is because, consider a quantum mechanical state containing  $n$  electrons and hence  $-n$  elementary charges, where  $n$  is an integer. Denote the number operator of the electrons as  $\hat{N}$  and  $\hat{N}$  acting on the state gives eigenvalue  $n$ .

$$\hat{N}|\psi\rangle = n|\psi\rangle \quad (2)$$

Now if we apply transformation  $e^{i\hat{N}\theta}$  to the state, the state remains invariant up to a global phase factor

$$e^{i\hat{N}\theta}|\psi\rangle = e^{in\theta}|\psi\rangle \quad (3)$$

Therefore,  $|\psi\rangle$  transforms as a 1D representation of the circle group  $e^{i\hat{N}\theta}$  labeled by  $n$  where  $n$  is the number of (negative) charges contained in the state. Similarly, we would call irreps of other symmetry groups as symmetry charges.

In general, whenever a system has a fixed number of particles (the particles could be electrons, atoms, ions, etc. independent of how much electronic charge they carry), the system has the symmetry. This is usually called the 'charge conservation symmetry' in physics. If the charge conservation symmetry is broken, i.e. the system does not have a fixed number of particles, there are very serious consequences. The system will be a superconductor, a superfluid or a Bose-Einstein condensate.

As the irreps are one dimensional, their character is simply given by  $\chi^{(n)}(\theta) = e^{in\theta}$ . The set of characters, as continuous functions of  $\theta$ , satisfy the following orthogonality condition

$$\langle \chi^{(n_1)}, \chi^{(n_2)} \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-in_1\theta} e^{in_2\theta} = \delta_{n_1, n_2} \quad (4)$$

Among all the irreps, only the ones labeled by  $n = 1$  and  $n = -1$  are faithful. All others are unfaithful. Under tensor product, the set of irreps form a group – the group of integers.

$$D^{(n_1)} \otimes D^{(n_2)} = D^{(n_1+n_2)} \quad (5)$$

Q: what irrep does the 2D real orthogonal representation contain?

## 6 Applications of finite groups

A. Zee, Group theory in a Nutshell for Physicists, chapter III.2

### 6.1 Vibration of coupled oscillators

Now let's see how our understanding of symmetry allows us to obtain insight into the oscillation eigenmodes of certain systems without knowing all the details of the dynamics of the system.

Imagine that we have a mechanical system where several mass blocks are bound together and interact with each other. Suppose that the equilibrium configuration and the interaction of these mass blocks have certain symmetry. Then if they undergo small vibrations around their equilibrium position, the dynamics of the system also has the same symmetry. In particular, we can find elementary 'modes' of such vibrational motion which form irreducible representations of the symmetry. Let's see how this happens in the following examples.

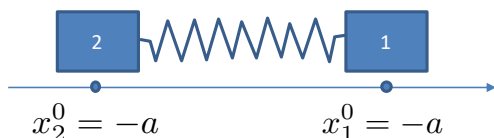


Figure 1: A coupled oscillator with reflection symmetry with respect to  $x = 0$ .

Let's consider a system composed of two identical blocks connected with a spring in the middle. The spring has Hooke constant  $k$ . At equilibrium, the two blocks are at  $x_1^0 = a$  and  $x_2^0 = -a$  respectively, as shown in Fig.1. This equilibrium configuration is invariant under reflection with respect to  $x = 0$ . Moreover, the form of interaction (generated by the spring) also respects this symmetry. What does this say about the dynamics of the system? From simple analysis (or purely physical intuition), we know that there are two eigenmodes for the motion of the two blocks. One is the center of mass motion, where the two blocks move together without changing their relative position. The other is relative motion, where the center of mass (the middle point of the central spring) does not move while the two blocks move relative to each other. These two modes of motion both transform in a special way under reflection. In the center of mass motion, the displacement are the same for block 1 and 2, therefore reflection maps the configuration to minus itself. In the relative motion, the displacement of the two blocks are opposite to each other, therefore, the configuration remains invariant under reflection. Let's try to describe this in a more concrete way.

The motion of the oscillator is described by  $x_1(t)$  and  $x_2(t)$ , the displacement of the two blocks relative to their equilibrium position. The equation of motion is given by

$$\begin{aligned} m \frac{d^2 x_1}{dt^2} &= -k(x_1 - x_2) \\ m \frac{d^2 x_2}{dt^2} &= -k(x_2 - x_1) \end{aligned} \quad (6)$$

Or in matrix form, we can write

$$m \frac{d^2 X}{dt^2} = -K X \quad (7)$$

where  $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ ,  $K = \begin{pmatrix} k & -k \\ -k & k \end{pmatrix}$ .

The eigenmodes are the vibrational motions which have a sin or cos dependence on time. That is, we are looking for solutions of the form  $X = \tilde{X} \cos(\omega t)$ . Plug it into the above equation we get

$$m\omega^2 \tilde{X} = K \tilde{X} \quad (8)$$

Therefore,  $\tilde{X}$ 's are eigenvectors of  $K$  with eigenvalue  $m\omega^2$ . Now, without knowing the details of  $K$ , it seems we cannot move forward. However, because we have the knowledge about the symmetry of the system, we can understand a lot about these  $\tilde{X}$ 's without actually diagonalizing  $K$ .

Recall that the system is invariant under reflection across  $x = 0$ . This corresponds to the linear transformation of  $D(b) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ . The fact that the dynamics of the system is invariant under this symmetry translates into the relation that

$$D^{-1}(b)KD(b) = K \quad (9)$$

or equivalently

$$KD(b) = D(b)K \quad (10)$$

What can we learn from this condition? Let's think in a group theoretical way. Notice that the reflection operation  $b$  generates a symmetry group. In this case, it is the  $C_2$  group with one other element  $e$ . Of course, there can be more general cases where we have a bigger symmetry group, but let's focus on this simple example first. The matrices  $D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  and  $D(b) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$  form a representation of the  $C_2$  group. Moreover, this representation is reducible. From our discussion in the last lecture, we know that with basis transformation generated by  $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ , we can put this reducible representation into a block diagonal form

$$HD(e)H^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, HD(b)H^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (11)$$

That is, this reducible representation contains both irreps of  $C_2$  and each irrep is contained once. The two irreps are supported on the one dimensional vectors  $H^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \propto \begin{pmatrix} 1 \\ 1 \end{pmatrix}$  and  $H^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \propto \begin{pmatrix} 1 \\ -1 \end{pmatrix}$  respectively. That is,  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$  each form a closed space under the symmetry transformation. On  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ , the group action is represented as  $D(e) = 1$ ,  $D(b) = -1$ ; on  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ , the group action is represented as  $D(e) = 1$ ,  $D(b) = 1$ .

On the other hand, we have the condition that  $D^{-1}(g)KD(g) = K$ , or equivalently  $KD(g) = D(g)K$ . From Schur's lemma, we can show that  $K$  has to take a block diagonal form in the irrep basis with each block being proportional to identity. That is

$$HKH^{-1} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad (12)$$

In order to show this, let's suppose that  $HKH^{-1} = \begin{pmatrix} a & c \\ d & b \end{pmatrix}$ . Because  $K$  is symmetric and  $H$  is orthogonal, we can set  $c = d$ . Then  $KD(g) = D(g)K$  translates into

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \begin{pmatrix} D^{(1)} & 0 \\ 0 & D^{(2)} \end{pmatrix} = \begin{pmatrix} D^{(1)} & 0 \\ 0 & D^{(2)} \end{pmatrix} \begin{pmatrix} a & c \\ c & b \end{pmatrix} \quad (13)$$

Comparing the off diagonal entries we find  $cD^{(1)} = cD^{(2)}$ , which can only be satisfied when  $c = 0$  as  $D^{(1)}$  and  $D^{(2)}$  are inequivalent irreps. On the other hand, there is no constraint on  $a$  and  $b$ .

Eq. 12 tells us that the eigenvectors of  $K$  are  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$  respectively. Therefore, without knowing the exact form of  $K$ , but by simply considering the symmetry constraints on  $K$ , we can find all its eigenvectors. They correspond to the support space of the irreps contained in the representation  $D(g)$ . And this matches with our intuitive expectation:  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  corresponds to the center of mass motion where the displacement of 1 and 2 are the same;  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$  corresponds to the relative motion where the displacement of 1 and 2 are opposite to each other. Of course, without knowing exactly the form of  $K$ , we will not be able to determine the eigenvalues  $a$  and  $b$  which in this case correspond to the oscillation frequency of the eigenmodes.