6 Applications of finite groups

6.1 Coupled harmonic oscillator

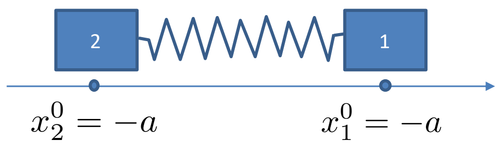

Figure 1: A coupled oscillator with reflection symmetry with respect to $x = 0$.

Let's briefly review what we did last time:

(1) We have a physical situation (coupled harmonic oscillator) where we need to solve an eigen equation $KX = \lambda X$.

(2) The system has some symmetry, which acts on X as matrices $D(g)$. The symmetry condition translates into $KD(g) = D(g)K$.

(3) Under certain basis transformation H, $D(g)$ can be put into a block diagonal form

$$HD(g)H^{-1} = \begin{pmatrix} D^{(1)}(g) & \\ & D^{(2)}(g) \end{pmatrix} \tag{1}$$

(4) If $D^{(1)}$ and $D^{(2)}$ are inequivalent one dimensional irreps, then under the same basis transformation, we have

$$HKH^{-1} = \begin{pmatrix} a & \\ & b \end{pmatrix} \tag{2}$$

(5) The eigenmodes of K are $H^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $H^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Each of them transforms under the symmetry as an irrep.

(6) There is not much we can say about the eigenvalues a and b which depends on the details of K.

In fact, this analysis applies not just to the particular configuration in Fig.1. It applies whenever the two blocks are coupled in a way that is symmetric under reflection. For example, we can imagine connecting the two blocks to walls on the two sides with springs of the same Hooke constant (which can be different from the middle block), as shown in Fig.2.

In this setup, the center of mass motion and the relative motion are still eigenmodes of this coupled oscillator, but they are going to have different oscillation frequency than in the previous case. In particular, now the center of mass motion is going to have a nonzero oscillation frequency, as compared to the previous case where the center of mass motion is not oscillating.

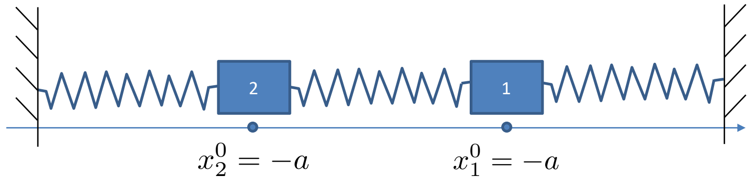

Figure 2: Another coupled oscillator with reflection symmetry with respect to $x = 0$.

6.2 A more complicated coupled harmonic oscillator

In the previous example, we considered a coupled harmonic oscillator with two degrees of freedom and a reflection symmetry of group C_2. We find that the eigenmodes correspond to the support space of the irreps of C_2 so they are completely fixed by the symmetry while the eigenvalues cannot be determined from symmetry consideration alone.

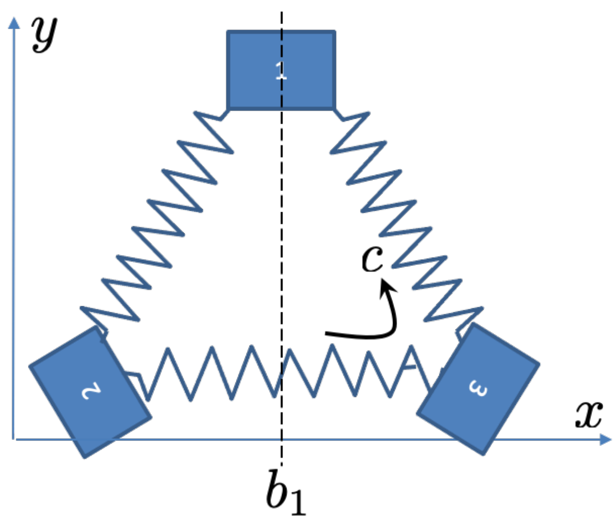

Figure 3: A coupled oscillator with permutation symmetry among 1, 2 and 3.

Now let's consider a slightly more complicated example of three blocks coupled together as shown in Fig.3, where we will run into the situation of higher dimensional irreps and multiple copies of equivalent irreps. Small (in plane) oscillation around this equilibrium position involves six dynamical degrees of freedom: the displacement of 1, 2 and 3 in x and y directions respectively $(x_1, y_1, x_2, y_2, x_3, y_3)$. The system has a D_3 symmetry involving three fold rotation and reflection. What can we tell about the oscillation eigenmodes from symmetry considerations?

First, let's try to see how this D_3 symmetry is represented on the six displacement coordinates. By working out how the displacement coordinates transform into each other, we find that the generators are represented as

$$D(c) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & -\sqrt{3} \\ 0 & 0 & 0 & 0 & \sqrt{3} & -1 \\ -1 & -\sqrt{3} & 0 & 0 & 0 & 0 \\ \sqrt{3} & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -\sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt{3} & -1 & 0 & 0 \end{pmatrix}, D(b_1) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad (3)$$

This is a reducible representation. The character of the representation is $\chi = \{6, 0, 0, 0, 0, 0\}$ from which we see that

$$D = D^{(1)} \oplus D^{(2)} \oplus 2D^{(3)} \tag{4}$$

($D^{(1)}, D^{(2)}, D^{(3)}$ as defined in the previous lectures.) That is, this six dimensional displacement space decomposes into one trivial irrep, one nontrivial $1D$ irrep and two copies of the $2D$ irrep. We can find a basis transformation S such that SDS^{-1} is in a block diagonal form with four blocks.

$$SD(g)S^{-1} = \begin{pmatrix} D^{(1)}(g) & & & \\ & D^{(2)}(g) & & \\ & & D^{(3)}(g) & \\ & & & D^{(3)}(g) \end{pmatrix} \tag{5}$$

WLOG, we will assume that the two $D^{(3)}$ blocks are completely the same (not just equivalent).

Note that because the last two blocks are the same, we can mix them by doing some basis transformation between these two blocks without changing the structure of the blocks. That is, we can apply transformation of the form $1 \oplus 1 \oplus (S_2 \otimes I_2)$ where S_2 is any 2×2 basis transformation. As $SD(g)S^{-1}$ is in the block diagonal form $D^{(1)}(g) \oplus D^{(2)}(g) \oplus (I_2 \otimes D^{(3)}(g))$, it remains invariant under the transformation

$$\begin{aligned} & (1 \oplus 1 \oplus (S_2 \otimes I_2)) \left(D^{(1)}(g) \oplus D^{(2)}(g) \oplus (I_2 \otimes D^{(3)}(g))\right) \left(1 \oplus 1 \oplus (S_2^{-1} \otimes I_2)\right) & (6) \\ = & D^{(1)}(g) \oplus D^{(2)}(g) \oplus (S_2 I_2 S_2^{-1} \otimes I_2 D^{(3)}(g) I_2) & (7) \\ = & D^{(1)}(g) \oplus D^{(2)}(g) \oplus (I_2 \otimes D^{(3)}(g)) & (8) \end{aligned}$$

What do these blocks correspond to? $D^{(1)}$ corresponds to the mode where the triangle shrink or expand as a whole, the corresponding displacement vector is $v_1 = \left(0, -1, \frac{\sqrt{3}}{2}, \frac{1}{2}, -\frac{\sqrt{3}}{2}, \frac{1}{2}\right)^T$. In particular, we can check that

$$D(g)v_1 = v_1, \ \forall g \in G \tag{9}$$

$D^{(2)}$ corresponds to the rotation of the triangle while expanding, the corresponding vector is $v_2 = \left(1, 0, -\frac{1}{2}, \frac{\sqrt{3}}{2}, -\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)^T$, which satisfies

$$D(c)v_2 = v_2, \ D(b_1)v_2 = -v_2 \tag{10}$$

The basis for the next two blocks are not uniquely fixed. But we can see that the center of mass motion corresponding to the space spanned by vectors $v_3 = (1, 0, 1, 0, 1, 0)^T$ and $v_4 = (0, 1, 0, 1, 0, 1)^T$ transform as $D^{(3)}$ while the remaining two dimensions transform as another $D^{(3)}$.

What does this tell us about the eigenmodes of the oscillator? The fact that the system has a D_3 symmetry implies that $D(g)K = KD(g)$. We are going to use Schur's lemma again to argue about the structure of K. In particular, let's suppose that under the basis transformation S, the matrix K takes the form

$$SKS^{-1} = \begin{pmatrix} a & b & A & B \\ b^* & c & C & D \\ A^\dagger & C^\dagger & E & F \\ B^\dagger & D^\dagger & F^\dagger & G \end{pmatrix} \tag{11}$$

where a, b, c are numbers, A, B, C, D are 1×2 matrices and E, F, G are 2×2 matrix. Because $D(g)$ and K commute, so do $SD(g)S^{-1}$ and SKS^{-1}. Therefore, we have

$$\begin{aligned} &\begin{pmatrix} D^{(1)}(g) & & & \\ & D^{(2)}(g) & & \\ & & D^{(3)}(g) & \\ & & & D^{(3)}(g) \end{pmatrix} \begin{pmatrix} a & b & A & B \\ b^* & c & C & D \\ A^\dagger & C^\dagger & E & F \\ B^\dagger & D^\dagger & F^\dagger & G \end{pmatrix} \\ = &\begin{pmatrix} a & b & A & B \\ b^* & c & C & D \\ A^\dagger & C^\dagger & E & F \\ B^\dagger & D^\dagger & F^\dagger & G \end{pmatrix} \begin{pmatrix} D^{(1)}(g) & & & \\ & D^{(2)}(g) & & \\ & & D^{(3)}(g) & \\ & & & D^{(3)}(g) \end{pmatrix} \end{aligned} \tag{12}$$

The left hand side is equal to

$$\begin{pmatrix} aD^{(1)}(g) & bD^{(1)}(g) & D^{(1)}(g)A & D^{(1)}(g)B \\ b^*D^{(2)}(g) & cD^{(2)}(g) & D^{(2)}(g)C & D^{(2)}(g)D \\ D^{(3)}(g)A^\dagger & D^{(3)}(g)C^\dagger & D^{(3)}(g)E & D^{(3)}(g)F \\ D^{(3)}(g)B^\dagger & D^{(3)}(g)D^\dagger & D^{(3)}(g)F^\dagger & D^{(3)}(g)G \end{pmatrix} \tag{13}$$

and the right hand side is equal to

$$\begin{pmatrix} aD^{(1)}(g) & bD^{(2)}(g) & AD^{(3)}(g) & BD^{(3)}(g) \\ b^*D^{(1)}(g) & cD^{(2)}(g) & CD^{(3)}(g) & DD^{(3)}(g) \\ A^\dagger D^{(1)}(g) & C^\dagger D^{(2)}(g) & ED^{(3)}(g) & FD^{(3)}(g) \\ B^\dagger D^{(1)}(g) & D^\dagger D^{(2)}(g) & F^\dagger D^{(3)}(g) & GD^{(3)}(g) \end{pmatrix} \tag{14}$$

Now we are going to use Schur's lemma which states that if $AD^{(\mu)}(g) = D^{(\nu)}(g)A$ for inequivalent irreps μ and ν, then $A = 0$ and if $AD^{(\mu)}(g) = D^{(\mu)}(g)A$, $A \propto I$. We get $b = 0$, $A = B = C = D = 0$, $E, F, G \propto I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Therefore,

$$SKS^{-1} = \begin{pmatrix} a & & & \\ & c & & \\ & & eI_2 & fI_2 \\ & & f^*I_2 & gI_2 \end{pmatrix} \tag{15}$$

where a, c, e, f, g are numbers.

We can see that the support space of the two one dimensional irreps each form an eigenmode of the coupled oscillator.

We still have four dimensions left, corresponding to the two copies of the two dimensional irrep. In the four dimensions, SKS^{-1} takes the form $\begin{pmatrix} e & f \\ f^* & g \end{pmatrix} \otimes I_2$. We can further perform a basis transformation and diagonalize the $\begin{pmatrix} e & f \\ f^* & g \end{pmatrix}$ part. If we denote the total basis transformation as S', then

$$S'KS'^{-1} = \begin{pmatrix} a & & & \\ & c & & \\ & & e'I_2 & \\ & & & g'I_2 \end{pmatrix} \tag{16}$$

Therefore, we conclude that in this four dimensional space, K can be decomposed into two diagonal blocks, each of two dimensions. Symmetry transformation on each of the two dimensional subspaces form an irrep $D^{(3)}$ and the two dimensions have the same eigenvalue.

□

Let's try to summarize what we learned from these examples.

(1) We have a physical problem which can be reduced to solving an eigenvalue equation of a Hermitian matrix K: $KX = \lambda X$.

(2) The system has certain symmetry and the degrees of freedom transform under the symmetry as $D(g),\ g \in G$. The matrix K satisfies $D(g)KD(g)^{-1} = K$.

(3) If we decompose $D(g)$ into irreducible blocks, we have $D(g) = \oplus n_\mu D^{(\mu)}(g)$.

Then we can make the following conclusions:

(4) Suppose that for some irrep μ of dimension d_μ, $n_\mu = 1$. The corresponding d_μ dimensional vector space forms a degenerate subspace of eigenvectors. All the vectors in this vector space are eigenvectors and they all have the same eigenvalue.

(5) For some other irrep μ of dimension d_μ, n_μ may be larger then 1. In the corresponding $n_\mu d_\mu$ dimensional vector space, we can find a number of n_μ degenerate subspaces, each of dimension d_μ. From symmetry considerations alone, we cannot determine the basis for each subspace.

(6) From symmetry considerations alone, we cannot determine the eigenvalue of each degenerate subspace. Among the degenerate subspaces, some of them may share the same eigenvalue, but this is not guaranteed by symmetry and is said to be accidental.

(7) The more symmetry the system has, the more constraint we can put on the eigenvectors. For example, suppose that with symmetry G vectors X^a and X^b belong to equivalent irreps while with symmetry $G'(G \subset G')$ they belong to inequivalent irreps. Therefore, with symmetry G' we can conclude that X^a and X^b cannot be mixed to form an eigenvector while with symmetry G alone we cannot make the statement.

To summarize: The eigenvectors of K can be grouped into irreps. Eigenvectors in the same irrep must be degenerate (have the same eigen-frequency) while eigenvectors in different irreps (which may or may not be equivalent to each other) generically have different eigen-frequency. A short take home message is: Degenerate eigen-spaces transform as irreps under the symmetry group.