7 Continuous Group

7.2 $SO(3)$

Irreps

Let's see how this looks like in some simple cases.

First, consider the case of $j = 0$. This irrep is one dimensional and the operators are all represented as numbers

$$L_z = 0, L_x = 0, L_y = 0, L^2 = 0 \tag{1}$$

If we exponentiate them, the rotation operators we get are all trivial

$$R_{\vec{n}}(\theta) = e^{-i\theta(n_x L_x + n_y L_y + n_z L_z)} = 1 \tag{2}$$

Next, let's move on to the case of $j = 1$. This irrep is three dimensional with basis states $|1\rangle$, $|0\rangle$, $|-1\rangle$. In matrix form, L_x, L_y, L_z and L^2 read

$$L_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; L_x = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}, \; L_y = \frac{i}{2}\begin{pmatrix} 0 & -\sqrt{2} & 0 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}, \; L^2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{3}$$

We know of another three dimensional representation of $SO(3)$ which is given by the three dimensional special orthogonal matrices. How are these two representations related? If we list the generators of the special orthogonal matrices, we can see that

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4}$$

This is related to the L_x, L_y, L_z given above by a basis transformation

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & i & 0 \end{pmatrix} \tag{5}$$

Therefore, the two three dimensional representations we have seen so far, are equivalent to each other.

Similarly, we can build up representations of five, seven, ... dimensions. Each of them correspond to a different irrep of $SO(3)$. That is, $SO(3)$ has one equivalence class of irrep in every odd dimension, labeled by integer j.

Characters

For an irrep labeled by j as derived previously, we can find the character of a conjugacy class labeled by θ by taking the trace of $R_z^j(\theta)$. In particular

$$J_z = \text{diag}(j, j-1, ..., -j) \tag{6}$$

Therefore

$$R_z^j(\theta) = \text{diag}(e^{ij\theta}, e^{i(j-1)\theta}, ..., e^{-ij\theta}) \tag{7}$$

and the character is

$$\chi^{(j)}(\theta) = \frac{\sin(j+1/2)\theta}{\sin(\theta/2)} \tag{8}$$

(take $j = 1$ and compare it to the case of the three dimensional special orthogonal representation.)

Are these characters orthogonal to each other? In order to answer the question, we need to define an integration for the group, which integrates over its parameter space. The parameter space of $SO(3)$ is highly nontrivial. First, we can specify every group element by a rotation axis direction $\vec{n}$ and an angle θ. $\vec{n}$ is a unit vector and we can take it to correspond to points on the surface of a unit ball. θ takes value from $-\pi$ to π and we can take it to correspond to the radial direction of the solid ball. However, the parameter space is not exactly the ball because a π rotation is the same as a $-\pi$ rotation. Therefore, the two ends of the same diameter should be identified. A solid ball with this identification gives the parameter space of $SO(3)$. The geometry and topology of this space is too complicated to discuss here. Instead I will just claim that the integration we want to use is

$$\int_0^{2\pi} \frac{d\theta}{2\pi}(1-\cos(\theta)) \tag{9}$$

and the characters are orthogonal under this integration

$$\langle \chi^{(j)}, \chi^{(j')} \rangle = \int_0^{2\pi} \frac{d\theta}{2\pi}(1-\cos(\theta)) \frac{\sin(j+1/2)\theta}{\sin(\theta/2)} \frac{\sin(j'+1/2)\theta}{\sin(\theta/2)} = \delta_{jj'} \tag{10}$$

7.3 $SU(2)$: the special unitary matrices of dimension two

Before I move on to talk about how the irreps of $SO(3)$ combine with each other (in direct product), I would like to digress and talk about $SU(2)$ first. $SU(2)$ is very similar to $SO(3)$ but also different in very important ways. It turns out that the irreps of $SO(3)$ is a subset of irreps of $SU(2)$ and when physicist study 'addition of angular momentum', what they really do is to study the direct product of irreps of $SU(2)$ instead of $SO(3)$. So let's first understand what $SU(2)$ is.

Instead of starting from the definition of $SU(2)$, let's start by considering the three Pauli matrices

$$\sigma_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{11}$$

It is easy to check that (1) they are Hermitian finite dimensional matrices (2) they satisfy the commutation rule $[\sigma_a, \sigma_b] = i\epsilon_{abc}\sigma_c$. It seems that they fulfill the requirement of being the infinitesimal generator of $SO(3)$. Actually, not quite. You may notice that once exponentiated, they do not quite give rise to the $SO(3)$ group. In particular, consider the 2π rotation around a particular axis, say z

$$R_z^{(1/2)}(2\pi) = e^{-i2\pi\sigma_z} = -I_2 \tag{12}$$

That is, 2π rotation is not exactly doing nothing. Instead it adds a global phase factor of -1.

Therefore, the group generated by σ_x, σ_y, and σ_z is not quite the $SO(3)$ group, in which doing 2π rotation should be the same as doing nothing. Instead, it generates the $SU(2)$ group: the group of special unitary matrices of dimension two.

Let's make linear superpositions of the infinitesimal generators and take their exponential.

$$R_{\vec{n}}(\theta) = e^{-i\theta\sigma_{\vec{n}}} = e^{-i\theta(n_x\sigma_x+n_y\sigma_y+n_z\sigma_z)} \tag{13}$$

Because $(\sigma_{\vec{n}})^2 = (n_x\sigma_x + n_y\sigma_y + n_z\sigma_z)^2 = I_2/4$, we have

$$R_{\vec{n}}(\theta) = \sum_{k=0}^{\infty} \frac{(-i\theta\sigma_{\vec{n}})^k}{k!} = \sum_{\text{even } k} \frac{(-i\theta/2)^k}{k!} + \sum_{\text{odd } k} \frac{(-i\theta/2)^k}{k!} 2\sigma_{\vec{n}} = \cos\left(\frac{\theta}{2}\right) - i\sin\left(\frac{\theta}{2}\right) 2\sigma_{\vec{n}} \tag{14}$$

This represents all possible 2×2 unitary matrices with determinant 1. (homework)

Although we started from three generators with the same commutation relation as those for $SO(3)$, the major difference between $SU(2)$ and $SO(3)$ is that the parameter takes value in $[0, 4\pi)$, not $[0, 2\pi)$. Only when $\theta = 4\pi$ does $R_{\vec{n}}(\theta)$ equal identity.

While $R_{\vec{n}}(2\pi)$ is not equal to identity, it is proportional to identity. Therefore, it commutes with all other group elements and generates the center of the group (recall the definition of the center), which is a C_2 group. As the center of a group is a normal subgroup as well, we can take the quotient of $SU(2)$ with respect to this C_2 group and we recover the $SO(3)$ group as the quotient group. We say that $SU(2)$ is a double cover of $SO(3)$.

You may wonder why we care about $SU(2)$ so much in physics. As it turns out, a very important property of electrons (and other fundamental particles) is their internal spin. This is not related to the orbital motion of the electron around a nucleus. Instead, it is something intrinsic to the electron. People realized that electron spin lives in a two dimensional Hilbert space. We can choose the basis state of this two dimensional Hilbert space as the eigenstates of σ_z.

$$\sigma_z\left|\frac{1}{2}\right\rangle = \frac{1}{2}\left|\frac{1}{2}\right\rangle, \sigma_z\left|-\frac{1}{2}\right\rangle = -\frac{1}{2}\left|-\frac{1}{2}\right\rangle \tag{15}$$

The raising and lowering operator maps between the two

$$\sigma_+ = \sigma_x + i\sigma_y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \sigma_- = \sigma_x - i\sigma_y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \sigma_+\left|-\frac{1}{2}\right\rangle = \left|\frac{1}{2}\right\rangle, \sigma_-\left|\frac{1}{2}\right\rangle = \left|-\frac{1}{2}\right\rangle \tag{16}$$

This is exactly the same relation as those given in the previous lecture if we set $j = \frac{1}{2}$. Therefore, this spin 1/2 behaves in every way like its integer angular momentum cousins, with one difference. Under spatial rotation around axis $\vec{n}$ through angle θ, it transforms as

$$R_{\vec{n}}(\theta) = e^{-i\theta\sigma_{\vec{n}}} = e^{-i\theta(n_x\sigma_x+n_y\sigma_y+n_z\sigma_z)} \tag{17}$$

which does not form a representation of the $SO(3)$ group but the $SU(2)$ group. This is a special property of quantum mechanics. That is, we can have quantum mechanical wave functions transforming under symmetry operations up to a phase factor. Here the phase factor shows up as $R_{\vec{n}}(2\pi) = -I$. This is ok in quantum mechanics because global phase factor is not measurable. It is in some sense a redundancy of the wave function representation, but through this example we can see that this redundancy is absolutely crucial because without it, there cannot be a spin 1/2 representation of rotation symmetry!

In fact, $SU(2)$ has other irreducible representations as well. All the irreps of $SO(3)$ are irreps of $SU(2)$, even though they are not quite faithful. Moreover, $SU(2)$ has one irrep in every even dimension which can be obtained in exactly the same way as the odd dimensional irreps for $SO(3)$ but starting from half integer j, $j = 1/2, 3/2, ...$ Of course, only the odd dimension irreps are irreps of $SO(3)$. The even dimension ones are irreps of $SO(3)$ only up to a phase factor, and we say that they are projective representations of $SO(3)$.

In physics, people often mix the notion of $SO(3)$ and $SU(2)$. It happens because in quantum mechanics both projective and nonprojective representations are allowed and they both show up in physical situations (like electron spin and orbital angular momentum) and can interact with each other.