9 Lorentz Group and Special Relativity

9.1 Lorentz Transformations

What kind of Λ satisfies the above condition?

First, any spatial rotation involving $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ keeps the length of the four vector invariant. Therefore, the spatial rotation transformations $\in SO(3)$ forms a subgroup of the Lorentz group. The transformation matrices take the form

$$\Lambda^r_{\vec{n}}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R_{\vec{n}}(\theta) & \\ 0 & & & \end{pmatrix} \tag{1}$$

where $R_{\vec{n}}(\theta)$ is the three dimensional special orthogonal matrix representing the spatial rotation. This subgroup of transformations is generated by

$$X_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2}$$

A different kind of Lorentz transformations which do involve time are the 'boosts'. Boost in the x direction gives rise to the transformation

$$\Lambda^b_x(v) = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3}$$

where $\gamma = \frac{c}{\sqrt{c^2 - v^2}}$. If we define $\gamma = \cosh\zeta = \frac{e^\zeta + e^{-\zeta}}{2}$, so that $\tanh\zeta = \frac{\sinh\zeta}{\cosh\zeta} = \frac{v}{c}$, then $\Lambda^b_x(v)$ can be re-written as

$$\Lambda^b_x(\zeta) = \begin{pmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4}$$

One can explicitly check that $\left(\Lambda^b_x(\zeta)\right)^T g \Lambda^b_x(\zeta) = g$.

The infinitesimal generator for x direction boost is

$$Y_1 = i\frac{d\Lambda^b_x(\zeta)}{d\zeta}\Big|_{\zeta=0} = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5}$$

By exponentiating Y_1, we can recover $\Lambda^b_x(\zeta)$

$$\Lambda^b_x(\zeta) = e^{-i\zeta Y_1} \tag{6}$$

However, notice one important difference with the generator for $SO(3)$: ζ is not bounded. As v approaches c, ζ approaches $+\infty$. Therefore, the Lorentz group $SO(3,1)$ is not compact. This has a series of consequences. One of them being that the finite dimensional representations of $SO(3,1)$ are no longer unitary.

It might seem that if we choose the parameter to be v/c which takes value between 0 and 1, then the parameter space would be compact. But v/c is not the right parameter to choose because if we define infinitesimal generator Y_1 with respect to v/c, we get the same Y_1 (verify this). But if we then try to recover the group elements by taking e^{-iv/cY_1}, we do not recover all the group elements.

Similarly, boosts in y and z directions are generated by

$$Y_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, Y_3 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \tag{7}$$

Boosts in an arbitrary direction $\vec{n}$ can be obtained first by rotating $\vec{n}$ to x axis, applying the boost in x direction and then rotating back.

In general, an arbitrary Lorentz transformation contains both spatial rotation and boost. The whole group is generated from X_1, X_2, X_3 and Y_1, Y_2, Y_3. The Lie algebra of $SO(3,1)$ is a six dimensional real vector space with commutators

$$[X_a, X_b] = i\epsilon_{abc}X_c, [X_a, Y_b] = i\epsilon_{abc}Y_c, [Y_a, Y_b] = -i\epsilon_{abc}X_c \tag{8}$$

Comments:

(1) While X_1, X_2, X_3 are closed under commutation, Y_1, Y_2, Y_3 are not. Therefore, the boosts do not form a subgroup.

(2) While X_1, X_2, X_3 are Hermitian, Y_1, Y_2, Y_3 are anti-Hermitian. Therefore, the representation is not unitary (the boost transformations are not unitary).

(3) The first and second commutation relation says that X_1, X_2, X_3 transform as a vector under $SO(3)$, so do Y_1, Y_2, Y_3.

(4) We can make linear combinations between X and Y

$$X_a^{(\pm)} = \frac{1}{2}(X_a \pm iY_a) \tag{9}$$

In terms of $X_a^{\pm}$, the commutation relations become

$$[X_a^{(+)}, X_b^{(+)}] = i\epsilon_{abc}X_c^{(+)}, [X_a^{(-)}, X_b^{(-)}] = i\epsilon_{abc}X_c^{(-)}, [X_a^{(+)}, X_b^{(-)}] = 0 \tag{10}$$

That is, the set of six generators break up into two subsets, such that each subset is equivalent to the Lie algebra of $SU(2)$ and the two subsets are independent of each other.

9.2 Irreducible representations

The four dimensional matrices Λ provide one possible representation of $SO(3,1)$ while the group is abstractly defined as that satisfying the same composition rule as the Λ's. In terms of Lie algebra, the group is defined as that with a six dimensional Lie algebra, satisfying the commutation relation

$$[X_a, X_b] = i\epsilon_{abc}X_c, [X_a, Y_b] = i\epsilon_{abc}Y_c, [Y_a, Y_b] = -i\epsilon_{abc}X_c \tag{11}$$

or

$$[X_a^{(+)}, X_b^{(+)}] = i\epsilon_{abc}X_c^{(+)}, [X_a^{(-)}, X_b^{(-)}] = i\epsilon_{abc}X_c^{(-)}, [X_a^{(+)}, X_b^{(-)}] = 0 \tag{12}$$

Now we can try to see what irreducible representations $SO(3,1)$ have. Following our analysis of $SO(3)$, in order to find irreps for a Lie group, we can try to find the irreps for its Lie algebra, but with the danger that we get the irrep of the covering group ($SU(2)$ for $SO(3)$). Things work in a very similar way for $SO(3,1)$.

We see that the Lie algebra of $SO(3,1)$ contains two $SU(2)$ part. Therefore, its irrep can be labelled by (j_1, j_2), where j_1, j_2 are integer or half-integer. The representation is then $(2j_1+1)(2j_2+1)$ dimensional. The generators are

$$X_a^{(+)} = J_a^{j_1} \otimes I_{2j_2+1}, X_a^{(-)} = I_{2j_1+1} \otimes J_a^{j_2} \tag{13}$$

From which we get

$$X_a = J_a^{j_1} \otimes I_{2j_2+1} + I_{2j_1+1} \otimes J_a^{j_2}, Y_a = -i(J_a^{j_1} \otimes I_{2j_2+1} - I_{2j_1+1} \otimes J_a^{j_2}) \tag{14}$$

If we then take the exponential, we can recover the group (or its covering group).

Let's see some example irreps.

(1) $j_1 = 0$, $j_2 = 0$

This is the trivial representation. It is one dimensional, with all the generators being 0 and all the group elements being represented by 1. In quantum field theory, this representation is carried by a relativistic scalar field (e.g. the Higgs field).

(2) $j_1 = 1/2$, $j_2 = 0$

This is called a spinor representation. It is two dimensional.

$$X_1^+ = \sigma_x, X_2^+ = \sigma_y, X_3^+ = \sigma_z, X_1^- = 0, X_2^- = 0, X_3^- = 0 \tag{15}$$

Correspondingly

$$X_1 = \sigma_x, X_2 = \sigma_y, X_3 = \sigma_z, Y_1 = -i\sigma_x, Y_2 = -i\sigma_y, Y_3 = -i\sigma_z \tag{16}$$

The Lorentz transformations are then parameterized by six real numbers θ_1, θ_2, θ_3, ϕ_1, ϕ_2, ϕ_3

$$\Lambda(\vec{\theta}, \vec{\phi}) = e^{i(\vec{\theta}\cdot\vec{X} + \vec{\phi}\cdot\vec{Y})} = e^{i(\vec{\theta} - i\vec{\phi})\cdot\vec{\sigma}} \tag{17}$$

Note that this is different from the $SU(2)$ group which contains matrices $e^{i\vec{\theta}\cdot\vec{\sigma}}$. In fact, the set of matrices generated are the group of special (determinant 1) linear (invertible) matrices of dimension

2, $SL(2,\mathbb{C})$. $SL(2,\mathbb{C})$ is a covering group of the $SO(3,1)$ group as 2π spatial rotation results in $-I$ instead of I.

In quantum field theory, this representation is carried by the Weyl fermion.

(3) $j_1 = 0$, $j_2 = 1/2$

This is another spinor representation with generators

$$X_1 = \sigma_x, X_2 = \sigma_y, X_3 = \sigma_z, Y_1 = i\sigma_x, Y_2 = i\sigma_y, Y_3 = i\sigma_z \tag{18}$$

This is inequivalent to the previous representation because they cannot be related by a basis transformation.

In quantum field theory, the direct sum of the $j_1 = 1/2$, $j_2 = 0$ representation and the $j_1 = 0$, $j_2 = 1/2$ representation is carried by the Dirac fermion.

(4) $j_1 = 1/2$, $j_2 = 1/2$

This is a four dimensional representation. Actually, it is exactly the four dimensional representation which we used to define $SO(3,1)$. That is, the space time four vector transforms as this representation.

10 Group Theory and Standard Model

Group theory played a big role in the development of the Standard model, which explains the origin of all fundamental particles we see in nature. In order to understand how that works, we need to learn about a new Lie group: $SU(3)$.

10.1 $SU(3)$ and more about Lie groups

$SU(3)$ is the group of special ($\det U = 1$) unitary ($UU^\dagger = I$) matrices of dimension three. What are the generators of $SU(3)$? If we want three dimensional matrices X such that $U = e^{i\theta X}$ is unitary (eigenvalues of absolute value 1), then X need to be Hermitian (real eigenvalue). Moreover, if U has determinant 1, X has to be traceless. Therefore, the generators of $SU(3)$ are the set of traceless Hermitian matrices of dimension 3. Let's count how many independent parameters we need to characterize this set of matrices (what is the dimension of the Lie algebra). 3×3 complex matrices contains 18 real parameters. If it is to be Hermitian, then the number of parameters reduces by a half to 9. If we further impose traceless-ness, then the number of parameter reduces to 8. Therefore, the generator of $SU(3)$ forms an 8 dimensional real vector space.

We can choose a basis for this eight dimensional vector space as

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{19}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{20}$$

They are called the Gell-Mann matrices. Among them λ_3 and λ_8 are diagonal and the other six correspond to the off-diagonal part of the Hermitian matrices. If we recall the structure of the Lie algebra of $SU(2)$, λ_3 and λ_8 are similar to J_z and others are similar to J_x and J_y.

By convention, we define

$$T_a = \frac{1}{2}\lambda_a \tag{21}$$

Note that T_1, T_2 and T_3 are exactly the σ_x, σ_y and σ_z operators acting on the first two dimensions. Therefore, they form a $su(2)$ sub-algebra and generate an $SU(2)$ subgroup of $SU(3)$.

The eight basis infinitesimal generators are ortho-normal in the sense that

$$Tr(T_a T_b) = \frac{1}{2}\delta_{ab} \tag{22}$$

From the T_a's we can determine the commutator of the Lie algebra

$$[T_a, T_b] = i f_{abc} T_c \tag{23}$$

where f_{abc} is called the structure constant of the group and is completely antisymmetric with respect to the exchange of any two indices. Recall for $SU(2)$, the structure constant was ϵ_{abc} which has a similar antisymmetric property.

The three dimensional special unitary matrices provide only one possible representation of $SU(3)$. This is called the fundamental representation. How to find the other representations? We can proceed in a similar way as $SU(2)$. Remember that for $SU(2)$, what we did was

1. find the Casimir operator which commute with the whole algebra and use its eigenvalue to label different representations

2. within each representation, use the eigenstates of J_z to label different basis states

3. define the raising and lowering operators $J_\pm$ to map from one basis state to another and determine the largest and smallest J_z eigenvalue given j.

Let's try to do something similar for $SU(3)$. First, the $su(3)$ algebra has two Casimir operators

$$C_1 = \sum_{i=1}^{8} T_i^2, C_2 = \sum_{ijk} d_{ijk} T_i T_j T_k \tag{24}$$

where d_{ijk} can be obtained from the anti-commutation relation for the generators

$$\{T_i, T_j\} = T_i T_j + T_j T_i = \frac{1}{3}\delta_{ij} + d_{ijk} T_k \tag{25}$$

Direct calculation shows that $C_1 = \frac{4}{3}$, $C_2 = \frac{10}{9}$ for the fundamental representation. In fact, the value for C_1 and C_2 can be obtained from two integers p and q.

$$C_1 = \frac{1}{3}(p^2 + q^3 + 3p + 3q + pq), \; C_2 = \frac{1}{18}(p-q)(3+p+2q)(3+q+2p) \tag{26}$$

Therefore, instead of using C_1 and C_2 to label representations, we can use p and q and denote each irrep as $D(p,q)$. The fundamental representation is then labeled as $D(1,0)$.

Now we want to find a set of basis states for each irrep. Similar to $SU(2)$, we can use the eigenstates of T_3 as basis states. In fact, because T_8 commute with T_3, we can use their common eigenstates as basis states of the irrep. There are no other independent generator which commutes with both. Of course, we can make other choices of generators to define basis states, but it can be shown that different choices are all equivalent to each other and give the same result.

In a Lie algebra, the subset of commuting hermitian generators which is as large as possible is called the Cartan subalgebra. We can choose a basis for this subalgebra H_i such that

$$H_i = H_i^\dagger, [H_i, H_j] = 0, Tr(H_iH_j) = k_D\delta_{ij}(k_D \text{ is representation and normalization dependent}) \tag{27}$$

The H_i, $i = 1, ..., m$, are called the Cartan generators and m is called the rank of the algebra.

The basis states of an irreducible representation can then be labeled by eigenvalues of H_i

$$H_i|p,q,\vec{\mu}\rangle = \mu_i|p,q,\vec{\mu}\rangle \tag{28}$$

In the case of fundamental representation of $SU(3)$, we have three basis states labeled by (apart from $p=1$, $q=0$)

$$\vec{\mu} = \left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right), \left(-\frac{1}{2}, \frac{\sqrt{3}}{6}\right), \left(0, -\frac{\sqrt{3}}{3}\right) \tag{29}$$

The vectors $\vec{\mu}$ are called the weight vectors of the algebra.

Now we can compose 'raising' and 'lowering' operators out of the other six generators of $su(3)$. Similar to $su(2)$, the raising and lowering operators can be chosen to have only one off-diagonal element. Define

$$E_{\pm 1,0} = \frac{1}{\sqrt{2}}\left(T_1 \pm iT_2\right), E_{\pm 1/2, \pm\sqrt{3}/2} = \frac{1}{\sqrt{2}}\left(T_4 \pm iT_5\right), E_{\mp 1/2, \pm\sqrt{3}/2} = \frac{1}{\sqrt{2}}\left(T_6 \pm iT_7\right) \tag{30}$$

Their subscript comes from their commutation relation with T_3 and T_8 and is related to how they change the weight vector. For example

$$[T_3, E_{\pm 1,0}] = \frac{1}{\sqrt{2}}\left([T_3, T_1] \pm i[T_3, T_2]\right) = (\pm 1)E_{\pm 1,0}, [T_8, E_{\pm 1,0}] = \frac{1}{\sqrt{2}}\left([T_8, T_1] \pm i[T_8, T_2]\right) = 0E_{\pm 1,0} \tag{31}$$

Correspondingly, application of $E_{\pm 1,0}$ to a state $|\vec{\mu}\rangle$ maps it to state $|\vec{\mu} + (\pm 1, 0)\rangle$. The vectors

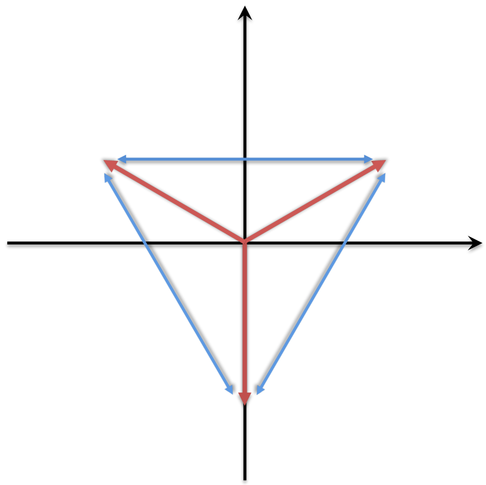
$$\vec{\alpha} = (\pm 1, 0), (\pm 1/2, \pm\sqrt{3}/2), (\mp 1/2, \pm\sqrt{3}/2) \tag{32}$$

are called roots of the algebra. Moreover, we can find

$$[E_{\vec{\alpha}}, E_{-\vec{\alpha}}] = \vec{\alpha} \cdot \vec{H} \tag{33}$$

which is a generalization of $SU(2)$ relation $[J_+, J_-] = 2J_z$ (if we rescale J_+, J_- by $\frac{1}{\sqrt{2}}$, we get $[J_+, J_-] = J_z$). Note that while the weight vectors (red) are specific to each representation, the root vectors (blue) are the same for all representations.

Now we can repeat the exercise of $SU(2)$ and try to find the set of weight vectors which make up a representation. In particular, for a particular p and q, we can start from a particular weight vector,



change it using the raising and lowering operators. By calculating how the norm of the basis state changes under such mapping (as a function of p, q, $\vec{\mu}$ and $\vec{\alpha}$), we can find the 'highest weight vector' and the 'lowest weight vector' and hence the dimension of the representation and the action of all generators in this basis. In this way, we can determine all irreducible representations of $SU(3)$. However, this calculation if too complicated and we are not going to do it explicitly.

Instead, we only mention here a few important representation of $SU(3)$. It turns out the dimension of a particular representation $D(p,q)$ is

$$d(p,q) = \frac{1}{2}(p+1)(q+1)(p+q+2). \tag{34}$$

1. $D(0,0)$

The dimension of this representation is 1 and this is the trivial representation. Every generator is represented as 0 and every group element is represented as 1. Still this is a very important representation in physics and it is called the singlet representation.

2. $D(1,0)$

This is the 3 dimensional representation given by the special unitary matrices. It is called the fundamental representation. We have discussed a lot about this representation above.

3. $D(0,1)$

This is again 3 dimensional. The infinitesimal generators are related to those in $D(1,0)$ by $T'_a = -T^*_a$ and the group elements are related by complex conjugation.

4. $D(1,1)$

This representation is 8 dimensional. This is an important representation called the adjoint representation. The adjoint representation is a representation of a Lie group on the vector space of its Lie algebra. The $SU(3)$ group has 8 generators, therefore, its adjoint representation is 8 dimensional. The adjoint representation is obtained by interpreting the commutation relation

$$[\hat{T}_a, T_b] = if_{abc}T_c \tag{35}$$

as the action of $\hat{T}_a$ on the basis T_b of the Lie algebra, mapping them to different linear combinations of the basis. For example, because

$$[T_1, T_2] = iT_3, [T_1, T_3] = -iT_2, [T_1, T_4] = i\frac{1}{2}T_7, [T_1, T_7] = -i\frac{1}{2}T_4, [T_1, T_5] = -i\frac{1}{2}T_6, [T_1, T_6] = i\frac{1}{2}T_5 \tag{36}$$

Therefore, T_1 is represented by the 8×8 matrix

$$T_1^{\text{adjoint}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{i}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{i}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{i}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{37}$$

We can see that if we change the basis of the Lie algebra to that of the Carton operator and the raising and lowering operator, the matrix corresponding to T_3 and T_8 becomes diagonal with the diagonal vector being the root of the algebra.