10 Group Theory and Standard Model

10.2 Gauge Theory

Electromagnetic field, Interaction with electrons

Before we present the standard model, we need to explain what a gauge symmetry is.

The idea of gauge symmetry started from the study of electromagnetism. Recall that Maxwell's equation reads

$$\nabla \cdot \vec{B} = 0,\ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t},\ \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0},\ \nabla \times \vec{B} = \mu_0 \vec{j} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \tag{1}$$

The first equation is automatically satisfied if

$$\vec{B} = \nabla \times \vec{A} \tag{2}$$

because $\nabla \cdot (\nabla \times \vec{A}) = 0$ for any $\vec{A}$. Substituting in the second equation we get

$$\nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0 \tag{3}$$

which can be automatically satisfied if

$$\vec{E} = -\nabla \varphi - \frac{\partial \vec{A}}{\partial t} \tag{4}$$

because $\nabla \times (\nabla \varphi) = 0$ for any φ.

The important observation here is that: given $\vec{E}$ and $\vec{B}$, $\vec{A}$ and φ are not uniquely determined. Instead, we get the same $\vec{E}$ and $\vec{B}$ if we change $\vec{A}$ and φ as

$$\vec{A} \to \vec{A} + \nabla f, \varphi \to \varphi - \frac{\partial f}{\partial t} \tag{5}$$

here f can be any real function of space and time. This is called a gauge transformation of the electromagnetic potentials φ and $\vec{A}$ and φ and $\vec{A}$ are called the gauge fields. Under the gauge transformation, the $\vec{E}$ and $\vec{B}$ fields remain invariant, hence the Maxwell's equation remains invariant. Therefore, the gauge transformation is a symmetry of the Maxwell's equation. Note that the gauge transformation is different from a global symmetry transformation in that the transformation can be different at different space time point. In this sense, it is called a local symmetry of E&M.

Now , we can re-write the last two Maxwell's equations using φ and $\vec{A}$ and try to solve for them, while keeping in mind that any two solution which differ by

$$(\varphi', \vec{A}') = (\varphi, \vec{A}) + (-\frac{\partial f}{\partial t}, \nabla f) \tag{6}$$

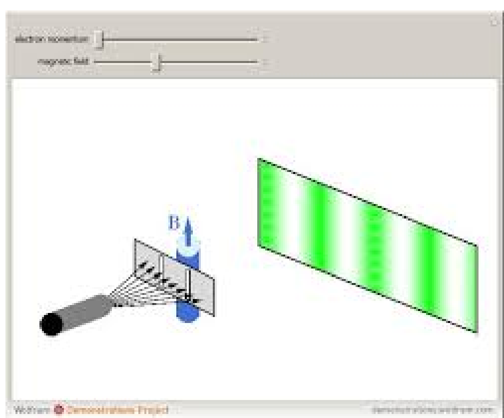
are equivalent solutions for the EM field.

Recall that we saw in homework 8 that the electromagnetic field $(\vec{E}, \vec{B})$ forms a representation of the Lorentz symmetry. This is now easier to see if we realize that $(\varphi, \vec{A})$ forms a four vector representation of Lorentz symmetry. The $(\vec{E}, \vec{B})$ is then a rank two tensor representation of Lorentz symmetry because it is obtained by taking derivative of a four vector $(\varphi, \vec{A})$ with respect to another four vector $(t, \vec{r})$. (Recall the definition of tensor from lecture 10.) Note that the gauge symmetry we have been talking about is independent from the Lorentz symmetry and the $(\varphi, \vec{A})$ fields transform under both.

Interaction with electrons

But why do we want to use the $(\varphi, \vec{A})$ fields to study electromagnetism if they are redundant? Why not just use the $(\vec{E}, \vec{B})$ fields? It may seem that $(\varphi, \vec{A})$ are just objects that we made up while $(\vec{E}, \vec{B})$ are the real physical fields. However, it was realized that $(\varphi, \vec{A})$ are indeed real (even though they are redundant).

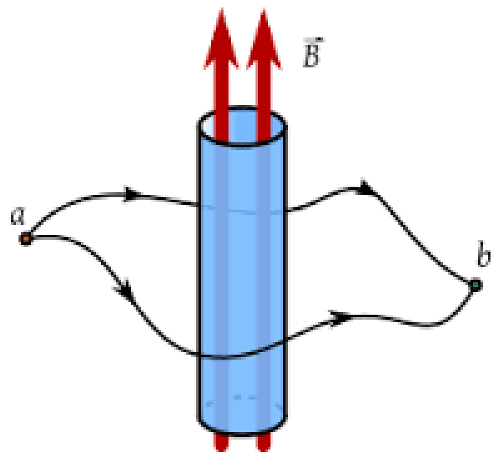
It took a lot of effort and a lot of confusion and what people realized was that we need to write our theory with $(\varphi, \vec{A})$ instead of $(\vec{E}, \vec{B})$ if we are to keep the theory local. This comes from the observation of the Aharonov-Bohm effect, where electron wave function undergoes a phase shift if the electron moves around a solenoid (manifested as a shift of interference pattern in a double slit experiment). A solenoid has nonzero magnetic field $\vec{B}$ only on its inside and zero magnetic field on the outside, so if we are to describe this effect in terms of the interaction of the electron with the $\vec{B}$ field, the interaction becomes nonlocal. On the other hand, the $\vec{A}$ field is nonzero even outside the solenoid, therefore, the AB effect can be attributed to the local interaction between the electron and $\vec{A}$. We introduced gauge redundancy in order to recover locality of the theory.



In particular, the equation of motion for an electron in an electromagnetic field is given by

$$i\hbar \frac{\partial \psi}{\partial t} = \left(\frac{\left(\vec{p} - e\vec{A}\right)^2}{2m} + e\varphi \right) \psi \tag{7}$$

Without the electromagnetic field, the $\frac{p^2}{2m}$ term describes the kinetic energy of the electron. The $e\varphi$ term describes the electric potential experienced by the electron. By changing $\vec{p}$ to $\vec{p} - e\vec{A}$ we couple the electron to the magnetic field as well and in classical mechanics one can check that this is the correct form for the Hamiltonian to reproduce the Lorentz force ($\vec{F} = e\vec{v} \times \vec{B}$) experienced by the electron. For quantum mechanics, we take this form of Hamiltonian and make position and momentum into operators.



Therefore, the basic equation describing the motion of an electron in an electromagnetic field is written in terms of $(\varphi, \vec{A})$ instead of $(\vec{E}, \vec{B})$. If $\psi(\vec{r}, t)$ is the wave function satisfying the Schroedinger equation without the $\vec{A}$ field, then the wave function satisfying the Schroedinger equation with the $\vec{A}$ field is

$$\psi'(\vec{r}, t) = e^{i\phi}\psi(\vec{r}, t), \quad \phi = \frac{e}{\hbar}\int_0^{\vec{r}} \vec{A}(\vec{r}') \cdot d\vec{r}' \tag{8}$$

That is, the $\vec{A}$ field changes the phase factor of the electron wave function which leads to the shift of the interference pattern.

But $(\varphi, \vec{A})$ is redundant. How does the gauge transformation of $(\varphi, \vec{A})$ change the equation? If we replace $(\varphi, \vec{A})$ with

$$\vec{A}' = \vec{A} + \nabla f, \varphi' = \varphi - \frac{\partial f}{\partial t} \tag{9}$$

The equation becomes

$$i\hbar\frac{\partial \psi}{\partial t} = \left(\frac{\left(\vec{p} - e\vec{A} - e\nabla f\right)^2}{2m} + e\varphi - e\frac{\partial f}{\partial t}\right)\psi \tag{10}$$

It looks like a different equation, but these extra terms of gauge transformation can actually be absorbed if we re-define

$$\psi' = \mathbf{e}^{ief/\hbar}\psi \tag{11}$$

where f is the real function on space time.

$$(\vec{p} - e\nabla f)\psi' = -i\hbar\mathbf{e}^{ief/\hbar}\nabla\psi + e\nabla f\mathbf{e}^{ief/\hbar}\psi - e\nabla f\mathbf{e}^{ief/\hbar}\psi = -i\hbar\mathbf{e}^{ief/\hbar}\nabla\psi \tag{12}$$

therefore the right hand side becomes

$$\mathbf{e}^{ief/\hbar}\left(\frac{\left(\vec{p} - e\vec{A}\right)^2}{2m} + e\varphi - e\frac{\partial f}{\partial t}\right)\psi \tag{13}$$

While the left hand side becomes

$$i\hbar\frac{\partial \psi'}{\partial t} = i\hbar\mathbf{e}^{ief/\hbar}\frac{\partial \psi}{\partial t} - e\frac{\partial f}{\partial t}\mathbf{e}^{ief/\hbar}\psi \tag{14}$$

Comparing the term we see that we get back to the original equation

$$i\hbar\frac{\partial\psi}{\partial t} = \left(\frac{\left(\vec{p} - e\vec{A}\right)^2}{2m} + e\varphi\right)\psi \tag{15}$$

Therefore, the Schrodinger's equation is invariant under the following transformation

$$\vec{A}' = \vec{A} + \nabla f, \varphi' = \varphi - \frac{\partial f}{\partial t}, \psi' = \mathbf{e}^{ief/\hbar}\psi \tag{16}$$

The transformation on the wave function of ψ is by adding a phase factor that is space and time dependent. Therefore, it is a local $U(1)$ transformation. The $(\varphi, \vec{A})$ field is correspondingly called a $U(1)$ gauge field. The phase factor change to ψ is proportional to e, the charge carried by the electron. If we have a particle with charge $2e$, the phase factor involved in the transformation would double.

The way we put EM fields into the equation for an electron is with the following changes:

$$i\hbar\frac{\partial}{\partial t} \to i\hbar\frac{\partial}{\partial t} - e\varphi,\ -i\hbar\frac{\partial}{\partial\vec{r}} \to -i\hbar\frac{\partial}{\partial\vec{r}} - e\vec{A} \tag{17}$$

This is called minimal coupling.

Nonabelian gauge field

While the concept of a gauge field may already look highly nontrivial, we need to generalize it to nonabelian gauge groups in order to formulate the theory underlying standard model. In particular, we need not only $U(1)$ gauge field, but also $SU(2)$ and $SU(3)$ gauge field.

Recall that a particle coupled to the $U(1)$ gauge field transform under the gauge transformation as

$$\psi \to \mathbf{e}^{in\theta}\psi \tag{18}$$

Here we are setting constants e and $\hbar$ as 1. n is the charge carried by the particle. Or in terms of representation theory, n labels the irrep of $U(1)$ carried by the particle. The wave function of the particle then transforms under the gauge transformation as the corresponding irrep labeled by n. The group element of the transformation, labeled by θ here, is space time dependent.

If a particle is coupled to a $SU(2)$ gauge field, it first needs to form a representation of $SU(2)$ labeled by j. If $j > 0$, then the wave function has $(2j + 1)$ components. Then under $SU(2)$ gauge transformation, the wave function transforms

$$\psi \to U(g(x,t))\psi = \mathbf{e}^{i\sum_{k=1}^{3}\theta_k J_k}\psi \tag{19}$$

Here ψ is a $2j+1$ component vector, J_k's are the three generators of $SU(2)$ in the irrep labeled by j, θ_k's label the group element of the transformation and can be space time dependent.

The gauge field of $SU(2)$ takes value in its Lie algebra and can be written as a linear combination of the generators.

$$W = W^k J_k \tag{20}$$

(The gauge field of $U(1)$ has only one component corresponding to its only one generator.) Note that similar to the $U(1)$ gauge field A, each W also has a μ label, where $\mu = 0, 1, 2, 3$. Moreover, these fields can have space time dependence (through the space time dependence of W^k).

The gauge fields then transform under the gauge transformation as

$$W_\mu \to U W_\mu U^{-1} - i(\partial_\mu U)U^{-1} \tag{21}$$

Notice how this reduces to the gauge transformation for the $U(1)$ gauge field if W and U is one dimensional. (Here we have used the metric $(-1, 1, 1, 1)$ so that $\partial_0 = -\partial/\partial t$, $\partial_1 = \partial/\partial x$, $\partial_2 = \partial/\partial y$, $\partial_3 = \partial/\partial z$.)

Then if in the Hamiltonian the particle wave function and the gauge field couple as

$$\partial_\mu \to \partial_\mu - iW_\mu \tag{22}$$

Then the form of the Schrodinger's equation remains invariant (homework).

We said that the particle forms an irrep labeled by j. What representation does the gauge field form for $SU(2)$? This is a strange question because the gauge field transforms under the gauge transformation which is space time dependent, so in principle the transformation group is $SU(2)$ to the power of number of space time points. But if we assume we apply the same transformation to all space time points, i.e. W^k has no space time dependence, then the gauge symmetry becomes a global symmetry and the symmetry group is $SU(2)$. Then we can see that the gauge field transformations as

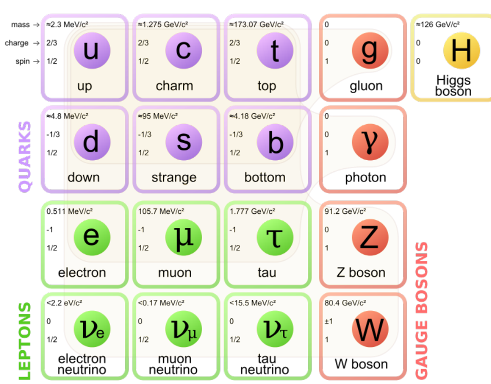
$$W_\mu \to U W_\mu U^{-1} \tag{23}$$

which is exactly how the Lie algebra transforms under the group. Therefore, the gauge field forms the adjoint representation of the group.

All of the discussion for $SU(2)$ can be straight forwardly generalized to the case of $SU(3)$. The gauge field has then 8 components and transforms as the adjoint representation of $SU(3)$ while the particle coupled to it can form an arbitrary representation labeled by p, q.

10.3 Standard Model

How does all this relate to the particles and forces of the Standard Model? The standard model, in its most commonly accepted formulation, contains a $U(1)$ gauge field, a $SU(2)$ gauge field, and a $SU(3)$ gauge field. The fundamental particles, like the quarks and the electron, couple to these gauge fields. They transform under the gauge symmetries as certain representations of the $SU(3) \times SU(2) \times U(1)$ group.



The elementary particles in the standard model fall into two major classes: gauge bosons and fermions. The Quarks and Leptons are fermions, with half integer spin and the Gauge bosons are bosons, with integer spin.

The gauge bosons come from the quantization of the gauge fields. That is, we can take the gauge fields, find their Fourier modes, quantize them so that each Fourier mode becomes a quantum harmonic oscillator like degree of freedom. The gauge bosons are then the quantized excitations of these harmonic oscillators. There is one type of gauge boson corresponding to every dimension of the Lie algebra of the gauge group.

For $SU(3)$, its Lie algebra is eight dimensional. Correspondingly, there are eight gauge bosons called the 'Gluons'. For the remaining $SU(2) \times U(1)$ gauge group, the Lie algebra is four dimensional, corresponding to one photon, two types of W boson and one type of Z boson. Note that photon does not correspond to the Lie algebra of the $U(1)$ part of the gauge group. Instead, the four generators of $SU(2) \times U(1)$ mix together and form some linear combinations that correspond to the photon and the W and Z bosons. The $SU(2) \times U(1)$ part is called the 'electroweak' subgroup of the total gauge group.

The other type of elementary particles are the fermions. The fermions form some spinor representation of the Lorentz group (with half integer spins) and at the same time form some representation of the gauge group. The quarks forms a nontrivial (the fundamental) representation of $SU(3)$. The three components of the quark field are called 'red', 'green', and 'blue' – the color charge of the quarks. The quarks hence couple to the $SU(3)$ gauge field and can interact with each other by exchanging gluons. The force mediated by the $SU(3)$ gauge field is called the strong interaction.

Leptons, on the other hand, do not transform under $SU(3)$ and therefore couple only to the electroweak part of the gauge group and experience only electroweak interaction. Among them the electron, muon and tau carry electromagnetic charge while the neutrinos are neutral. The neutrinos only interact through the weak interaction.

Finally, there is the Higgs boson, which has a different origin. The Higgs boson is responsible for the symmetry breaking of the electroweak part of the gauge group and gives mass to the W and Z bosons.