Physics 223b **Homework 1** **Due 04/26/17 by 5pm**

1. Bogoliubov approach to superfluidity.

In this problem, we are going to derive the linear dispersion relation of a superfluid using Bogoliubov's approach. Consider the Bose-Hubbard model in d-dimensional continuous space with Hamiltonian

$$H = \int d\mathbf{x} \left(\frac{-\hbar^2}{2m} \psi^\dagger_{\mathbf{x}} \nabla^2 \psi_{\mathbf{x}} \right) - \mu \psi^\dagger_{\mathbf{x}} \psi_{\mathbf{x}} + U \int d\mathbf{x}_1 d\mathbf{x}_2 \rho_{\mathbf{x}_1} \rho_{\mathbf{x}_2} \delta(\mathbf{x}_1 - \mathbf{x}_2) \tag{1}$$

where $\psi^\dagger$ and ψ are the boson creation and annihilation field operators satisfying $[\psi_{\mathbf{x}}, \psi^\dagger_{\mathbf{x}'}] = \delta(\mathbf{x} - \mathbf{x}')$. $\rho_{\mathbf{x}} = \psi^\dagger_{\mathbf{x}} \psi_{\mathbf{x}}$ is the density operator at $\mathbf{x}$.

1. Apply the Fourier transform to the boson field operators and define

$$\psi_{\mathbf{k}} = \frac{1}{\sqrt{V}} \int d\mathbf{x} e^{-i\mathbf{k}\mathbf{x}} \psi_{\mathbf{x}}, \quad \psi^\dagger_{\mathbf{k}} = \frac{1}{\sqrt{V}} \int d\mathbf{x} e^{i\mathbf{k}\mathbf{x}} \psi^\dagger_{\mathbf{x}} \tag{2}$$

   Use the relation $\int d\mathbf{x} e^{-i\mathbf{k}\mathbf{x}} = V \delta_{\mathbf{k}0}$ to determine the commutator $[\psi_{\mathbf{k}}, \psi^\dagger_{\mathbf{k}'}]$.

2. Rewrite the Hamiltonian using $\psi_{\mathbf{k}}$ and $\psi^\dagger_{\mathbf{k}}$.

3. Close to the ground state, the interaction term is dominated by terms involving $\psi_{\mathbf{k}=0}$. Suppose that in the ground state $\psi_{\mathbf{k}=0} = \gamma$. As long as $|\gamma| \gg 1$, we can treat $\psi^\dagger_{\mathbf{k}=0}$ as a number as well and its value is γ^*.

   The largest term in the interaction is then

$$\psi^\dagger_{\mathbf{k}=0} \psi_{\mathbf{k}=0} \psi^\dagger_{\mathbf{k}=0} \psi_{\mathbf{k}=0} = |\gamma|^4 \tag{3}$$

   The second largest terms are

$$4\psi^\dagger_{\mathbf{k}=0} \psi_{\mathbf{k}=0} \psi^\dagger_{\mathbf{k}} \psi_{\mathbf{k}} + \psi^\dagger_{\mathbf{k}=0} \psi^\dagger_{\mathbf{k}=0} \psi_{\mathbf{k}} \psi_{-\mathbf{k}} + \psi_{\mathbf{k}=0} \psi_{\mathbf{k}=0} \psi^\dagger_{-\mathbf{k}} \psi^\dagger_{\mathbf{k}} \tag{4}$$

$$= 4|\gamma|^2 \psi^\dagger_{\mathbf{k}} \psi_{\mathbf{k}} + (\gamma^2)^* \psi_{\mathbf{k}} \psi_{-\mathbf{k}} + \gamma^2 \psi^\dagger_{-\mathbf{k}} \psi^\dagger_{\mathbf{k}} \tag{5}$$

   All other terms involve four nonzero momentums and are much smaller than the previous two. We can ignore them if we stay close to the ground sate.

   Combined with the kinetic and chemical potential term, the total Hamiltonian becomes

$$H_{\mathbf{k}=0} + \sum_{\mathbf{k}\neq 0} \left(\frac{\hbar^2 \mathbf{k}^2}{2m} - \mu + 4\frac{U}{V} |\gamma|^2 \right) \psi^\dagger_{\mathbf{k}} \psi_{\mathbf{k}} + \frac{U}{V} (\gamma^2)^* \psi_{-\mathbf{k}} \psi_{\mathbf{k}} + \frac{U}{V} \gamma^2 \psi^\dagger_{\mathbf{k}} \psi^\dagger_{-\mathbf{k}} \tag{6}$$

   For $\mathbf{k} \neq 0$, recombine $\psi^\dagger_{\mathbf{k}}$ and $\psi_{\mathbf{k}}$ into a new set of operators $\alpha_{\mathbf{k}}$

$$\alpha_{\mathbf{k}} = u_{\mathbf{k}} \psi_{\mathbf{k}} + v_{\mathbf{k}} \psi^\dagger_{-\mathbf{k}} \tag{7}$$

   Find the set of parameters $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ such that

   1. $\alpha_{\mathbf{k}}$ each represents a new boson mode so that $[\alpha_{\mathbf{k}}, \alpha^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k},\mathbf{k}'}$, $[\alpha_{\mathbf{k}}, \alpha_{\mathbf{k}'}] = 0$.

   2. The Hamiltonian can be diagonalized in this basis. That is, the Hamiltonian can be written as $H = \sum_{\mathbf{k}\neq 0} E_{\mathbf{k}} \alpha^\dagger_{\mathbf{k}} \alpha_{\mathbf{k}} + E_0$.

   Without loss of generality, you can take γ to be real.

4. How does $E_{\mathbf{k}}$ depend on $\mathbf{k}$ when $\mathbf{k}$ is small? (choose the parameter so that $E_{\mathbf{k}=0} = 0$.)

2. Roton excitation in a superfluid.

A real superfluid is more complicated than a weakly interacting boson gas. On the one hand, the interacting is pretty strong. Also the interaction can extend to a finite range, instead of being just a delta function. Suppose that the interaction takes the form of a square function $V(\mathbf{r} - \mathbf{r}') = U\text{rect}\left(\frac{x-x'}{2r_0}\right)\text{rect}\left(\frac{y-y'}{2r_0}\right)\text{rect}\left(\frac{z-z'}{2r_0}\right)$ where

$$\text{rect}(t) = 1 \text{ when } |t| < \frac{1}{2}, \text{rect}(t) = 0 \text{ when } |t| > \frac{1}{2} \tag{8}$$

Note that we have chosen a highly anisotropic form of the interaction potential to make sure the following computation is tractable. The qualitative feature of our result will remain if we choose an isotropic form of the interaction potential.

Let's see how the excitation spectrum changes in the superfluid.

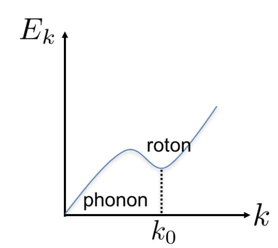
1. Repeat step 2 of the above problem to write the full Hamiltonian using the Fourier modes $\psi_{\mathbf{k}}$ and $\psi^\dagger_{\mathbf{k}}$. Use the relation

$$\int_{-\infty}^{\infty} \text{rect}(t) \cdot e^{-i2\pi f t}\, dt = \frac{\sin(\pi f)}{\pi f} = \text{sinc}(\pi f) \tag{9}$$

2. Use the approximation that $|\psi_{\mathbf{k}=0}| = |\gamma| \gg \psi_{\mathbf{k}\neq 0}$ to reduce the Hamiltonian to quadratic form, as in step 3 of the above problem.

3. Repeat step 3 of the above problem and write the Hamiltonian as $H = \sum_{\mathbf{k}\neq 0} E_{\mathbf{k}} \alpha^\dagger_{\mathbf{k}} \alpha_{\mathbf{k}} + E_0$. How does $E_{\mathbf{k}}$ depend on $|\mathbf{k}|$?

4. Plot $E_{\mathbf{k}}$ along the k_x direction for small $|k_x|$ and show that schematically it looks like the following figure.



   The excitations near $\mathbf{k} = 0$ has a linear dispersion and are called the phonon excitation in the superfluid. The excitations near $|\mathbf{k}| = k_0$ has a quadratic dispersion and are called the roton excitation in the superfluid.

5. Find the critical velocity when superfluidity breaks down. That is, find the minimum velocity when the viscosity of the fluid becomes nonzero. You may leave your result implicit, i.e. the critical velocity is a function $v = v(a)$ for dimensionless parameter a that solves the equation $f(a) = 0$.

3. Energy of a vortex – anti-vortex pair

In the lecture, we derived the relation

$$\mathbf{j} = \frac{\hbar}{m}\rho^2\nabla\theta \tag{10}$$

1. Use the relation $\mathbf{j} = n\mathbf{v}$ where n is the particle number density and v is average velocity to find that

$$\mathbf{v} = \frac{\hbar}{m}\nabla\theta \tag{11}$$

2. In a rotationally invariant vortex with vorticity k, show that

$$\mathbf{v} = \frac{\hbar k}{mr}\hat{e}_\phi \tag{12}$$

   where r is the distance from the vortex center and $\hat{e}_\phi$ is the unit vector in the angular direction.

3. The particle flow around a vortex is in precise analogy to the magnetic field in a wire carrying electrical current. Show that if we make the equivalence between $\mathbf{v}$ and $\mathbf{B}$, then the analog of vorticity k is given by Ampere's law

$$\mu_0 I = \oint \mathbf{B}\cdot d\mathbf{l} \tag{13}$$

4. Show that this analogy can be extended so that the kinetic energy per unit volume of superfluid

$$dE = \frac{1}{2}nmv^2 d^3\mathbf{r} \tag{14}$$

   is equivalent to the electromagnetic energy density of the wire

$$dE = \frac{1}{2\mu_0}B^2 d^3\mathbf{r} \tag{15}$$

   Hence finding exact equivalences between the physical parameters of superflow in a vortex $\{\frac{hk}{m}, nm, nm\mathbf{v}\}$ and the corresponding parameters for a wire $\{I, \mu_0, \mathbf{B}\}$

5. Given that the force per unit length between two parallel current carrying wires a distance R apart is

$$F = \frac{\mu_0 I_1 I_2}{2\pi R} \tag{16}$$

   Use the electromagnetic analogy above to find that the force per unit length between two parallel superfluid vortices with vorticity k_1, k_2 . Show that the energy needed to create a pair of vortex and anti-vortex from vacuum is given by

$$E = \frac{h^2 n}{2\pi m}\ln(R/R_0) \tag{17}$$

   where R_0 is a constant. This relation holds only when $R > R_0$.