Physics 223b **Homework 4** **Due 06/06/18 by 5pm**

1. Majorana chain with time reversal symmetry

Consider the Majorana chain in the topological phase

$$H = \sum_j i\gamma_{2j}\gamma_{2j+1} \tag{1}$$

where time reversal symmetry acts as

$$i \to -i, \quad \gamma_{2j-1} \to \gamma_{2j-1}, \quad \gamma_{2j} \to -\gamma_{2j} \tag{2}$$

Take M copies of the Majorana chain, so that on an open chain there are M Majorana modes at each end of the chain. In particular, at the left end, the Majoranan modes transform under time reversal as

$$\gamma_1^{(m)} \to \gamma_1^{(m)}, \quad m = 1, 2, ..., M \tag{3}$$

(1) Consider an increasing number of Majorana chains. What is the smallest number $M = M_0$ where the edge degeneracy can be completely removed by adding four fermion interaction terms (of the form $\gamma\gamma\gamma\gamma$) to the edge Majorana modes without breaking time reversal symmetry?

(2) Now consider the system on a ring with periodic boundary condition. For M_0 Majorana chains, design a gapped path of the Hamiltonian $H(s)$, $0 \le s \le 1$ so that $H(0)$ is the Hamiltonian of M_0 Majorana chains

$$H(0) = \sum_{m=1}^{M_0} \sum_j i\gamma_{2j}^{(m)}\gamma_{2j+1}^{(m)} \tag{4}$$

and $H(1)$ reads

$$H(1) = \sum_{m=1}^{M_0} \sum_j i\gamma_{2j-1}^{(m)}\gamma_{2j}^{(m)} \tag{5}$$

Show explicitly that the Hamiltonian is gapped for all $0 \le s \le 1$. (hint: consider a path through the point where on each lattice site, the M_0 Majorana fermions on the left hand side couple to each other through the four fermion interaction terms and the M_0 on the right hand side couple to each other in a similar way.)

2. Entanglement Area Law in Matrix Product States

Consider a finite dimensional matrix product state with open boundary condition

$$|\psi\rangle = \sum_{i_1 i_2 ... i_N} \langle l|A^{i_1}A^{i_2}...A^{i_N}|r\rangle |i_1 i_2 ... i_N\rangle \tag{6}$$

where $\langle l|$ and $|r\rangle$ are row and column vectors respectively. In this problem, we are going to show that the entanglement entropy between the left half of the chain (containing n lattice sites) and the right half of the chain (containing $N - n$ lattice sites) is upper bounded by a constant.

(1) Show that the wave function can be written as

$$|\psi\rangle = \sum_m |\psi_L^m\rangle|\psi_R^m\rangle \tag{7}$$

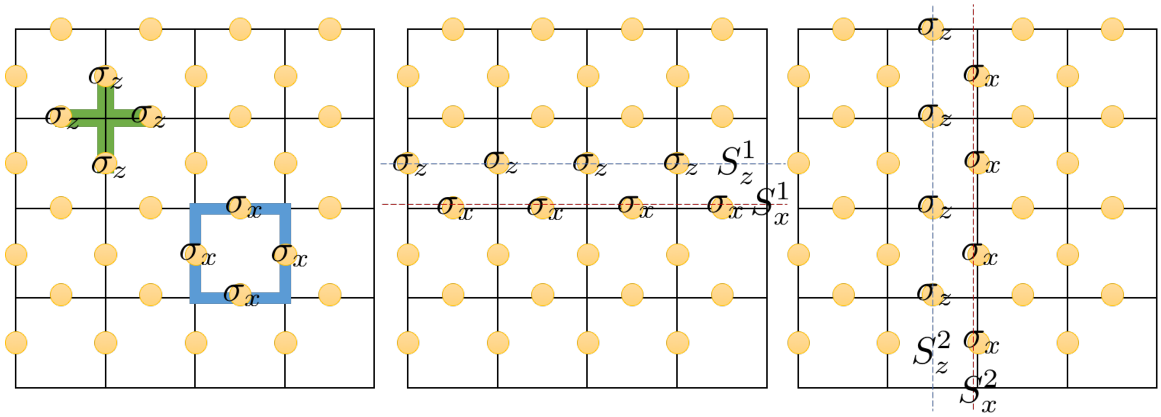
where m labels the inner dimensions of matrix A. Find the explicit form of $|\psi_L^m\rangle$ and $|\psi_R^m\rangle$.

(2) Show that the dimension of the reduced density matrix ρ_L (or ρ_R) is upper bounded by m.

(3) Entanglement entropy between the left and right part of the chain is given by $S = -\mathrm{Tr}\rho_L \ln \rho_L$. Show that S is upper bounded by a constant that depends on m.

3. Toric Code

In this problem, we are going to explore the basic properties of the Toric Code model in 2D. Consider a square lattice with periodic boundary condition and one spin 1/2 degree of freedom (basis states $|0\rangle$ and $|1\rangle$) per each link.



The Hamiltonian contains two types of terms: one involving four σ_z's around a vertex, one involving four σ_x's around a plaquette.

$$H = -\sum_v \left(\prod_{v\in l} \sigma_z^l\right) - \sum_p \left(\prod_{l\in p} \sigma_x^l\right) \tag{8}$$

σ_z and σ_x acts as

$$\sigma_z|0\rangle = |0\rangle, \quad \sigma_z|1\rangle = -|1\rangle, \quad \sigma_x|0\rangle = |1\rangle, \quad \sigma_x|1\rangle = |0\rangle \tag{9}$$

(a) Show that all the terms in the Hamiltonian commute with each other.

(b) As the Hamiltonian terms are all independent of each other, the ground state can be chosen to minimize energy of each of the Hamiltonian terms. We can think of the spin 1/2 degrees of freedom as a Z_2 string on each link. That is, the $|0\rangle$ state corresponds to no string on each link while the $|1\rangle$ state corresponds to the existence of a string on the link. To minimize the energy of the $-\prod_{v\in l}\sigma_z^l$ terms, what conclusion can you draw regarding the string configuration in the ground state? Draw three string configurations satisfying all the $-\prod_{v\in l}\sigma_z^l$ terms.

(c) Now within the subspace of states satisfying the $-\prod_{v\in l}\sigma_z^l$ terms, try to minimize the energy of all the $-\prod_{l\in p}\sigma_x^l$ terms. Write down the ground state wave function that contains the configuration which has no string (all $|0\rangle$ product state).

(d) For a square lattice on the torus, are there other ground states? What is the ground state degeneracy?

(e) Consider the string operator S_x^1, S_x^2, S_z^1, S_z^2 as shown in the figure. Show that they commute with the Hamiltonian. What is their action within the degenerate ground space?