Physics 223b **Homework 3** **Due 05/23/18 by 5pm**

1. Jordan Wigner Transformation of the Majorana Chain

Consider the Majorana chain with Hamiltonian

$$H = \sum_{j=1}^{N} -\mu c_j^\dagger c_j - t c_{j+1}^\dagger c_j - t c_j^\dagger c_{j+1} + \Delta c_{j+1}^\dagger c_j^\dagger + \Delta^* c_j c_{j+1} \tag{1}$$

Take Δ to be real and $t = \Delta$.

Apply the Jordan Wigner transformation and map it to a spin 1/2 chain

$$c_j^\dagger = \prod_{l<j} \sigma_l^z (\sigma_j^x + i\sigma_j^y)/2, \quad c_j = \prod_{l<j} \sigma_l^z (\sigma_j^x - i\sigma_j^y)/2, \tag{2}$$

(a) What does the total fermion parity symmetry $P_f = \prod_j (2c_j^\dagger c_j - 1)$ map to under the transformation?

(b) What does the $-\mu c_j^\dagger c_j$ term map to?

(c) In the middle of the chain ($1 \leq j < N$), what does the $-\Delta c_{j+1}^\dagger c_j - \Delta c_j^\dagger c_{j+1} + \Delta c_{j+1}^\dagger c_j^\dagger + \Delta c_j c_{j+1}$ term map to?

(d) On an open chain, what is the total spin Hamiltonian? What is the ground state when $\Delta = 0, \mu > 0$? Is the global Z_2 symmetry spontaneously broken? Show this by calculating the correlation function of the order parameter (σ_x) in the symmetric ground state wave function.

(e) On an open chain, what are the ground states when $\mu = 0, \Delta < 0$? Do they break the global symmetry of the system? Which fermionic ground states do they correspond to in the Majorana chain model?

(f) On a closed chain, what does the boundary term $-\Delta c_1^\dagger c_N - \Delta c_N^\dagger c_1 + \Delta c_1^\dagger c_N^\dagger + \Delta c_N c_1$ map to? What is the total spin Hamiltonian? Is the ground state degenerate?

2. Chiral $p + ip$ superconductor in 2D

Consider a spinless two dimensional superconductor with mean field Hamiltonian

$$H = \sum_r [-t(c_r^\dagger c_{r+\hat{x}} + h.c.) - t(c_r^\dagger c_{r+\hat{y}} + h.c.) + \Delta(c_r^\dagger c_{r+\hat{x}}^\dagger + h.c.) + i\Delta(c_r^\dagger c_{r+\hat{y}}^\dagger + h.c.)] - \sum_r \mu c_r^\dagger c_r \tag{3}$$

defined on a square lattice of size $L \times L$ and periodic boundary condition in both $\hat{x}$ and $\hat{y}$ directions. Here L is the number of sites along one axis.

(a) Perform the Fourier transformation

$$c_r^\dagger = \frac{1}{L} \sum_k e^{ik\cdot r} c_k^\dagger, \quad c_r = \frac{1}{L} \sum_k e^{-ik\cdot r} c_k \tag{4}$$

and write the Hamiltonian in terms of c_k and $c_k^\dagger$ and find the pairing function Δ_k.

(b) Diagonalize the Hamiltonian and find the eigenmodes (Hint: Write the matrix as $H_k = \vec{d}\cdot\vec{\sigma}$). Is the Hamiltonian gapped if we have $-4|t| < \mu < 4|t|$? What about for μ outside this range? What happens at $\mu = \pm 4|t|$? (As the Hamiltonian is particle-hole symmetric, we only need to consider eigenmodes with positive eigenvalues.)

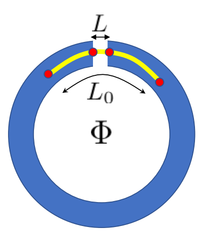
(c) Consider the normalized vector $\hat{d} = \frac{\vec{d}}{|d|}$, where $H_k = \vec{d}\cdot\vec{\sigma}$. When $\mu = 0$ and $t > 0$, how does $\hat{d}$ change with k? When $t = 0$ and $\mu > 0$, how does $\hat{d}$ change with k? Is there a topological difference between these two cases?

(d) Now consider the system with open boundary condition in the x direction but still periodic in the y direction. We can still do a Fourier transformation in the y direction but not in x direction. Find the form of the Hamiltonian H_{k_y} after the partial Fourier transform.

(e) For several values of parameters in the topological phase (e.g., $t = 1, \Delta = 0.75, \mu = 1$) and in the trivial phase (e.g., $t = 1, \Delta = 0.75, \mu = 6$), numerically solve for the eigenspectrum for each H_{k_y}. It is customary to visualize the spectrum by plotting k_y on the horizontal axis and energy on the vertical axis, and for each k_y showing the corresponding eigenvalues as a tower of points above k_y. You should be able to see the 'bulk band' appearing from points merging closer and closer as the system size grows, while in the topological phase you should be able to see two isolated 'edge bands' emerging. From the numerical eigenvectors, examine where the corresponding 'edge modes' reside (their distribution in x direction), while from the k_y dependence determine in which direction they are moving.

(f) Change the boundary condition in the y direction from periodic to anti-periodic. Re-plot the spectrum for both the topological and trivial phases. Do you see a change in zero energy modes in the edge band?

3. Detecting Majorana Zero Modes



Consider the configuration shown in the figure: on top of a superconducting ring (blue) with a Josephson Junction (white) is a semiconductor wire. When the semiconductor wire lies on top of the superconducting region, it is in the topological phase with Majorana zero modes at the ends (red dots). The existence of the Majorana zero modes can then be detected from the doubled periodicity of the critical current through the Josephson Junction with respect to the phase difference between the superconductors on the two sides. The phase difference ϕ can be controlled by putting a magnetic flux Φ through the ring and is given by $\phi = \Phi\frac{2\pi}{\Phi_0}$.

We can model the 1D wire as a left part and a right part coupled through a weak link. The Hamiltonian for the left part is:

$$H_L = \sum_{1\le j\le N} -\mu c_j^\dagger c_j - t c_{j+1}^\dagger c_j - t c_j^\dagger c_{j+1} + \sum_{1\le j\le N} \Delta c_j^\dagger c_{j+1}^\dagger + \Delta c_{j+1} c_j \tag{5}$$

The Hamiltonian for the right part is:

$$H_R = \sum_{N+1 \le j \le 2N} -\mu c_j^\dagger c_j - t c_{j+1}^\dagger c_j - t c_j^\dagger c_{j+1} + \sum_{N+1 \le j \le 2N} \Delta e^{i\phi} c_j^\dagger c_{j+1}^\dagger + \Delta e^{-i\phi} c_{j+1} c_j \tag{6}$$

Their coupling is given by a weak hopping term

$$H_c = -t' c_N^\dagger c_{N+1} - t' c_{N+1}^\dagger c_N \tag{7}$$

(a) Show that when $\mu = 0$, $t = \Delta$, H_L is in the topologically nontrivial phase with Majorana zero modes at the two ends. Identify the Majorana operator corresponding to the zero modes.

(b) Show that H_R can be mapped to the same form as H_L if we map c_j to $e^{i\phi/2} c_j$ and $c_j^\dagger$ to $e^{-i\phi/2} c_j^\dagger$. Identify the Majorana operator corresponding to the zero modes in H_R.

(c) Expand H_c in terms of the Majorana modes on the left and right hand side and keep only the term that couples the zero modes (because the coupling is weak). Plot the energy of the eigenstates of the reduced coupling term with respect to ϕ and find its period.

(d) Usually, the energy of a Josephson junction changes as a periodic function of ϕ with period 2π. For the Majorana wire Josephson junction, the period is doubled. Briefly explain why this is the case.