Physics 223b **Homework 2** **Due 05/09/18 by 5pm**

1. Boson Number Fluctuation in Superfluid.

Consider the superfluid phase whose ground state wave function is a coherent state

$$|\Psi_{SF}\rangle = \mathcal{N} e^{\sqrt{V}\gamma b^{\dagger}_{k=0}}|0\rangle \tag{1}$$

where $\mathcal{N}$ is the normalization factor, V is the total volume of the system, $|\gamma|^2 = n$ is the density of bosons in the system.

(a) Calculate the average value of the total boson number in the ground state $\langle N\rangle$, where N is $\int_r b^{\dagger}(r)b(r)$, and $b(r)$ and $b^{\dagger}(r)$ are the annihilation and creation operator of each spatial boson mode.

(b) Calculate the average value of the total boson number squared in the ground state $\langle N^2\rangle$.

(c) Calculate the fluctuation of the boson number $\langle \Delta N\rangle = \sqrt{\langle (N - \langle N\rangle)^2\rangle}$.

(d) How does $\langle \Delta N\rangle$ scale with $\langle N\rangle$ as $\langle N\rangle$ becomes large?

2. Type II Superconductor.

Consider a superconductor in a *large* magnetic field and close to the transition to the normal state so that the order parameter is close to zero. The magnetic field $\mathbf{B} = \mu_0\mathbf{H}$ points in the z direction $\mathbf{B} = (0, 0, B)$. Correspondingly, we can choose the vector potential to be

$$\mathbf{A} = (0, Bx, 0) \tag{2}$$

We are going to look for a solution to the Ginzburg Landau equation obtained by varying the free energy functional with respect to $\delta\psi$

$$-\frac{\hbar^2}{2m^*}\left(\nabla + \frac{2ei}{\hbar}\mathbf{A}\right)^2\psi + a\psi + b\psi|\psi|^2 = 0 \tag{3}$$

Plugging in the form of $\mathbf{A}$ and dropping the nonlinear term (due to the small-ness of the order parameter), we get

$$-\frac{\hbar^2}{2m^*}\left(\nabla + \frac{2eiBx}{\hbar}\hat{e}_y\right)^2\psi + a\psi = 0 \tag{4}$$

1. Show that solutions to this equation takes the form

$$\psi(\mathbf{r}) = e^{i(k_y y + k_z z)}f(x) \tag{5}$$

   such that $f(x)$ satisfies the equation

$$-\frac{\hbar^2}{2m^*}\frac{d^2 f}{dx^2} + \frac{m^*\omega_c^2}{2}(x - x_0)^2 f = \left(-a - \frac{\hbar^2 k_z^2}{2m^*}\right) f \tag{6}$$

   where $\omega_c = \frac{2eB}{m^*}$.

2. Using the similarity of Eq. 6 with the Hamiltonian of a simple harmonic oscillator, determine the eigenvalue $\left(-a - \frac{\hbar^2 k_z^2}{2m^*}\right)$. Show that $-a$ is lower bounded by $\frac{1}{2}\hbar\omega_c$.

3. At a fixed temperature $T < T_c$, start from a large external field so that the system is non-superconducting. Lower the external field so that the above equation can have at least one solution. What is the critical field strength H_{c2} when this happens? (Assume that $a = \dot{a}(T - T_c)$.)

4. Recall the definition of the coherence length $l_c(T) = \sqrt{\frac{\hbar^2}{2m^*|a|}}$. At the critical field, H_{c2}, how much flux is contained in each unit area $2\pi l_c^2(T)$?

5. Compare H_{c2} to the critical field $H_c = \sqrt{\frac{a^2}{\mu_0 b}}$ which was derived without considering the possibility of spatially varying order parameter field. Find the condition when we have an extra phase transition at H_{c2} before H_c, i.e., a type II superconductor. How is this condition related to the ratio between the London penetration depth

$$\lambda(T) = \sqrt{\frac{m^* b}{4\mu_0 e^2 |a|}} \tag{7}$$

and the coherence length $l_c(T)$?

3. BCS superconductor in Zeeman field

Consider the BCS Hamiltonian in the presence of a magnetic field B coupling only to the spin:

$$H = \sum_{\mathbf{k},\sigma} (\epsilon_{\mathbf{k}} - \mu - \sigma \mu_B B)\, c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k},\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'} c^\dagger_{\mathbf{k}'\uparrow} c^\dagger_{-\mathbf{k}'\downarrow} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \tag{8}$$

where $V_{\mathbf{k},\mathbf{k}'} = -V_0/\text{Vol}$ for $\mathbf{k}$, $\mathbf{k}'$ within a shell of energy width $\hbar\omega_D$ on either side of the Fermi surface and zero otherwise. μ_B is the Bohr magneton, $\sigma = +1$ or -1 for $\uparrow$ or $\downarrow$ spin respectively. (It is assumed that the superconductor as a 2D layer is thin enough to allow penetration of the magnetic field in the interior, and any orbital coupling is neglected.)

1. Show that the expectation value of H in the BCS wavefunction $|\Psi\rangle = \prod_{\mathbf{k}} u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}|0\rangle$ has the same energy as for $B = 0$. Hence, minimization of the energy with respect to the pair occupation amplitude gives the same result as at $B = 0$.

2. For small B, ignore the interaction term, calculate the lowering of the ground state energy of the free electron gas caused by the Zeeman coupling, to second order in B.

3. Combine 1 and 2 to obtain the reduction in the condensation energy of the superconductor caused by the Zeeman coupling. At what value of the magnetic field is superconductivity destroyed by the Zeeman coupling?

4. Calculate the minimum excitation energy (gap) of the BCS superconductor as a function of the field B at $T = 0$. When does the gap collapse? Compare the result with 3. This is so-called "Pauli limiting field", also known as "Chandrasekhar-Clogston limit".

5. Compare this value with that of the critical magnetic field obtained from coupling to the orbital magnetic moment. Make an estimate about which one is larger for a typical low temperature superconductor (you can look up relevant parameters in text books or online).