1 Simple Harmonic Oscillator

1.4 General properties of Simple Harmonic Oscillator

1.4.4 Superposition of two independent SHO

(to illustrate the usefulness of complex number)

Suppose we have two SHOs described by

$$x_1 = A_1 \cos(\omega_1 t + \varphi_1) = Re(A_1 e^{\omega_1 t + \varphi_1}) = Re(z_1) \tag{1}$$
$$x_2 = A_2 \cos(\omega_2 t + \varphi_2) = Re(A_2 e^{\omega_2 t + \varphi_2}) = Re(z_2) \tag{2}$$

What if the two motions are happening at the same time on the same degree of freedom? What does the total motion look like? Let's consider a couple of interesting situations.

A. $A_1 = A_2$, $\omega_1 = \omega_2$, but $\varphi_1 \neq \varphi_2$

To do the superposition of x_1 and x_2, we add z_1 and z_2 and take the real part

$$\begin{aligned} z_1 + z_2 &= A\left(e^{i(\omega t+\varphi_1)} + e^{i(\omega t+\varphi_2)}\right) \\ &= Ae^{i\omega t} e^{i(\varphi_1+\varphi_2)/2}\left[e^{i(\varphi_1-\varphi_2)/2} + e^{i(\varphi_2-\varphi_1)/2}\right] \\ &= Ae^{i(\omega t+\bar{\varphi})} 2\cos(\delta\varphi) = 2A\cos(\delta\varphi)e^{i\omega t+\bar{\varphi}} \\ x_1 + x_2 &= Re(z_1+z_2) = 2A\cos(\delta\varphi)\cos(\omega t+\bar{\varphi}) \end{aligned} \tag{3}$$

Therefore, the superposed motion is still of the simple sinusoidal form, with total amplitude $|2A\cos(\delta\varphi)|$. When $\delta\varphi = 0$, the total amplitude reaches its maximum of $2A$; the two SHOs are said to be in phase. When $\delta\varphi = \pi$, the total amplitude reaches its minimum of 0; the two SHOs are said to be out of phase and cancel each other. We say that when two SHOs have the same frequency, they can interfere, either constructively or destructively or somewhere in between.

Question: What happens if $\omega_1 = \omega_2$ but $A_1 \neq A_2$?

B. If $A_1 = A_2$, $\varphi_1 = \varphi_2$, $\omega_1 \neq \omega_2$, but $\omega_1 \approx \omega_2$

Something interesting happens in this situation. Suppose that at some time t, the two oscillations are in phase $\delta\varphi = 0$. On a short time scale, as the two have almost the same frequency, they interfere constructively. Some time later (on the scale of $1/(\omega_1 - \omega_2)$), the two oscillations fall out of phase and may even become completely out of phase with $\delta\varphi = \pi$. For some short period of time at this later point, the two would interfere destructively. Therefore, the total oscillation will alternate between very strong (large amplitude) and very weak (small amplitude) and this is the phenomena called 'Beat'.

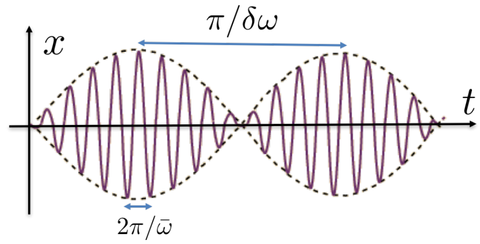
Mathematically, we use again the complex representation to do the superposition

$$\begin{aligned} z_1 + z_2 &= Ae^{i\omega_1 t} + Ae^{\omega_2 t} \\ &= Ae^{i(\omega_1+\omega_2)t/2}\left[e^{i(\omega_1-\omega_2)t/2} + e^{i(\omega_2-\omega_1)t/2}\right] \\ &= 2A\cos(\delta\omega t)e^{i\bar{\omega}t} \end{aligned} \tag{4}$$

where we have defined $\delta\omega = (\omega_1 - \omega_2)/2$ and $\bar{\omega} = (\omega_1 + \omega_2)/2$. Taking the real part, we have

$$x_1 + x_2 = 2A\cos(\delta\omega t)\cos\bar{\omega}t \tag{5}$$

which can be interpreted as oscillation at frequency $\bar{\omega}$, but with slowly time varying amplitude that changes with frequency $\delta\omega$, that is, a beat. We say that the high frequency oscillation is 'modulated' by the low frequency oscillation.

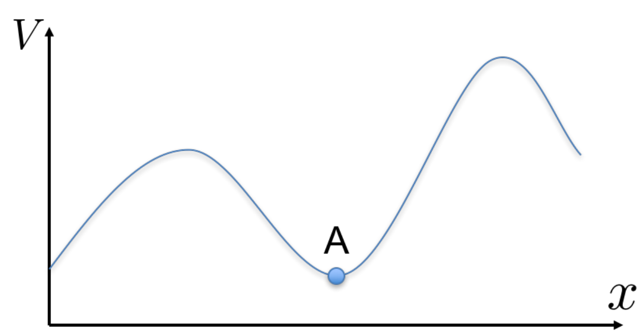


The phenomena of beat provides a very useful way to calibrate / measure frequency against a frequency standard.

Demo 20203: Frequency Generator Beat Pattern

1.4.5 Why is SHO so ubiquitous

The SHO describes all kinds of small oscillations around equilibrium configurations in a physical system. Consider a system with a potential landscape as shown below



Point A is a local minimum in the potential landscape and an equilibrium position. The potential gradient is zero at this point.

$$\left.\frac{dV}{dx}\right|_A = 0 \tag{6}$$

Close to A, the potential energy V can be Taylor expanded into

$$V(x) = V(A) + \frac{1}{2}\left.\frac{d^2V}{dx^2}\right|_A (x - x_A)^2 + ... \tag{7}$$

If the system stays close enough to point A, we can ignore the higher order terms in ... and the potential energy takes the same square form as the three examples given previously. In particular, the restoring force is given by

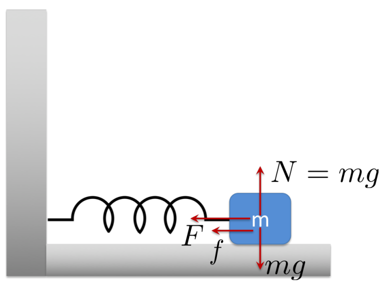
$$F = -\frac{dV(x)}{dx} = -\frac{d^2V}{dx^2}\Big|_A (x - x_A) \tag{8}$$

which is proportional to displacement and in the opposite direction. The resulting motion will be the sinusoidal oscillation around the equilibrium point.

2 Damped and Forced Harmonic Oscillator

2.1 Damped Harmonic Oscillator

Now let's consider the more realistic case where the system has dissipation so that the oscillatory motion cannot go on forever. We go back to the simple example of mass on the spring and consider now the situation where there is friction.



Friction exerts a force that's in the opposite direction of motion and takes a simple form of

$$-m\Gamma\frac{dx}{dt} \tag{9}$$

which is proportional to the mass of the block and its velocity. Γ is the friction constant. Note that it has the same dimension as ω. (Friction coming from the bottom surface does not take this form. Instead, this form of friction can come from, for example, fluids surrounding the moving body.)

The equation of motion now gets modified

$$m\frac{d^2}{dt^2}x(t) + m\Gamma\frac{d}{dt}x(t) + kx(t) = 0 \tag{10}$$

which can be reorganized into

$$\frac{d^2}{dt^2}x(t) + \Gamma\frac{d}{dt}x(t) + \omega_0^2 x(t) = 0 \tag{11}$$

Without solving the equation, we know that the solution should describe a decaying oscillation. Let's see what the math say.

To solve this equation for the real function $x(t)$, we solve a corresponding equation for a complex function $z(t)$

$$\frac{d^2}{dt^2}z(t) + \Gamma\frac{d}{dt}z(t) + \omega_0^2 z(t) = 0 \tag{12}$$

and find $x(t)$ by taking the real part.

We guess the form of the solution to be $z(t) = Ae^{-\alpha t}$, where α can in general be a complex number. We take A to be a complex number in general as well, although in many cases we will narrow down the choice of A later according to the physical situation under consideration.

Plugging this form of solution into the equation we find

$$\alpha^2 z(t) - \Gamma\alpha z(t) + \omega_0^2 z(t) = 0 \tag{13}$$

which leads to a quadratic equation for α

$$\alpha^2 - \Gamma\alpha + \omega_0^2 = 0 \tag{14}$$

The solution of α depends on the relationship between Γ and ω_0. Let's discuss the various cases.

(1) $\Gamma^2 > 4\omega_0^2$

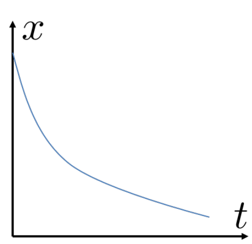
This is the case where friction is very large, and we would expect a pure decay of the oscillation.

The solution for α in this case is $\alpha_\pm = \frac{\Gamma \pm \sqrt{\Gamma^2 - 4\omega_0^2}}{2}$, which are two real positive numbers. The generic solution of $z(t)$ is given by

$$z(t) = A_+ e^{-\alpha_+ t} + A_- e^{-\alpha_- t} \tag{15}$$

When we take the real part, we find $x(t) = Re(z(t)) = Re(A_+)e^{-\alpha_+ t} + Re(A_-)e^{-\alpha_- t}$, which describes pure decay. We see that the imaginary part of A_+ and A_- does not enter the final solution, only the real part does, which can be determined from initial conditions.

Graphically, it looks something like this



This case is called over damping.

(2) $\Gamma^2 < 4\omega_0^2$

When friction is small, it should result in a gradual decay of the oscillation.

The solution for α in this case is $\alpha_\pm = \frac{\Gamma \pm i\sqrt{|\Gamma^2 - 4\omega_0^2|}}{2} = \frac{\Gamma}{2} \pm i\frac{\sqrt{4\omega_0^2 - \Gamma^2}}{2}$, which are complex numbers. The generic solution of $z(t)$ is given by

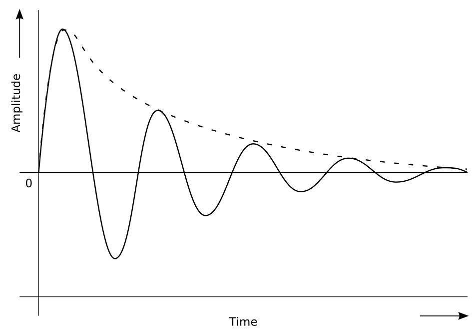
$$z(t) = A_+ e^{-\alpha_+ t} + A_- e^{-\alpha_- t} = A_+ e^{-\Gamma t/2} e^{i\omega t} + A_- e^{-\Gamma t/2} e^{-i\omega t} \tag{16}$$

where $\omega = \sqrt{\omega_0^2 - \Gamma^2/4} < \omega_0$. Taking the real part we get

$$x(t) = Re(z(t)) = Re(|A_+|e^{i\phi_+}e^{-\Gamma t/2}e^{i\omega t} + |A_-|e^{i\phi_-}e^{-\Gamma t/2}e^{-i\omega t}) \tag{17}$$

$$= |A_+|e^{-\Gamma t/2}\cos(\omega t + \phi_+) + |A_-|e^{-\Gamma t/2}\cos(\omega t - \phi_-) \tag{18}$$

which describes a decaying oscillation at frequency ω.



This case is called under damping.

In the general solution we have four free parameters $|A_+|$, $|A_-|$, ϕ_1, ϕ_2. But we know that we only have two initial conditions to fix these parameters, so the parameters are redundant. We can reorganize the terms into the form

$$x(t) = A_1e^{-\Gamma t/2}\cos(\omega t) + A_2e^{-\Gamma t/2}\sin(\omega t) \tag{19}$$

where $A_1 = |A_+|\cos(\phi_1) + |A_-|\cos(\phi_2)$, $A_2 = -|A_+|\sin(\phi_1) + |A_-|\sin(\phi_2)$ are two real parameters to be set by the initial conditions.

(3) $\Gamma^2 = 4\omega_0^2$

This is the case of critical damping and we will explore it in the homework.