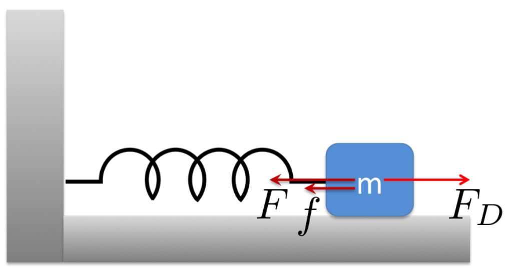
2 Damped and Driven Harmonic Oscillator

Demo 20103: Tuning Fork

2.2 Driven Harmonic Oscillation

In order to have a persistent oscillatory motion in a system with dissipation, we can drive it by pushing on the system periodically. Consider again the classic example of a mass on a spring, but now with a horizontal force $F_D(t)$ as the drive. Throughout the discussion of this section, we will assume that $\Gamma << \omega_0$. That is, if the system is not driven, it will undergo an oscillatory motion with slow decay.



Newton's law now reads

$$m\frac{d^2}{dt^2}x(t) = -kx(t) - m\Gamma\frac{d}{dt}x(t) + F_D(t) \tag{1}$$

Reorganizing the terms, we get

$$\frac{d^2}{dt^2}x(t) + \Gamma\frac{d}{dt}x(t) + \omega_0^2 x(t) = F_D(t)/m \tag{2}$$

where we have used $\omega_0^2 = k/m$.

Let's consider first the case where $F_D(t) = F_0\cos(\omega_D t)$ is a periodic force, but with a frequency ω_D that can be different from the intrinsic frequency ω_0. To solve the equation, we solve instead its complex version

$$\frac{d^2}{dt^2}z(t) + \Gamma\frac{d}{dt}z(t) + \omega_0^2 z(t) = \frac{F_0}{m}e^{i\omega_D t} \tag{3}$$

Let's try a solution of the form $z_D(t) = Ae^{i\omega_D t}$ which describes an oscillatory motion at the driving frequency. Plugging this solution into the equation, we get

$$-\omega_D^2 Ae^{i\omega_D t} + i\Gamma A\omega_D e^{i\omega_D t} + \omega_0^2 Ae^{i\omega_D t} = \frac{F_0}{m}e^{i\omega_D t} \tag{4}$$

From which we find

$$A = \frac{F_0}{m(\omega_0^2 + i\Gamma\omega_D - \omega_D^2)}, z_D(t) = \frac{F_0}{m(\omega_0^2 + i\Gamma\omega_D - \omega_D^2)}e^{i\omega_D t} \tag{5}$$

and

$$x_D(t) = Re(A)\cos(\omega_D t) - Im(A)\sin(\omega_D t) \tag{6}$$

That is, $x_D(t)$ has two oscillatory parts, one is in phase with $F_D(t)$ and the other is 90 degree out of phase. The amplitude of the in phase part is

$$|Re(A)| = \frac{F_0}{m}\frac{|\omega_0^2 - \omega_D^2|}{(\omega_0^2 - \omega_D^2)^2 + \Gamma^2\omega_D^2} \tag{7}$$

and the amplitude of the out of phase part is

$$|Im(A)| = \frac{F_0}{m}\frac{\Gamma\omega_D}{(\omega_0^2 - \omega_D^2)^2 + \Gamma^2\omega_D^2} \tag{8}$$

Let's try to understand what this solution means:

1. The amplitude of both parts are proportional to F_0, which makes sense. The larger the driving force, the larger the driven motion.

2. When $\omega_D << \omega_0$, $Im(A) \approx 0$ while $Re(A) \approx \frac{F_0}{m\omega_0^2}$. The driven motion is given by

$$x(t) \approx \frac{F_0}{m\omega_0^2}\cos(\omega_D t) \tag{9}$$

which is completely in phase with the driving force.

3. When $\omega_D >> \omega_0$ and $\omega_D >> \Gamma$, both $Re(A)$ and $Im(A)$ approaches zero, but $Im(A)$ goes to zero faster. So the driven motion is approximately given by

$$x(t) \approx -\frac{F_0}{m\omega_D^2}\cos(\omega_D t) = \frac{F_0}{m\omega_D^2}\cos(\omega_D t + \pi) \tag{10}$$

which is completely out of phase with the driving force.

4. Another important point in the parameter range of ω_D is when $\omega_0 = \omega_D$. At this point, $Re(A) = 0$, $Im(A) = -\frac{F_0}{m\Gamma\omega_D}$. The driven motion is given by

$$x(t) = \frac{F_0}{m\Gamma\omega_D}\sin(\omega_D t) = \frac{F_0}{m\Gamma\omega_D}\cos(\omega_D t - \frac{\pi}{2}) \tag{11}$$

That is, the driven motion is 90 degree out of phase with the driving force.

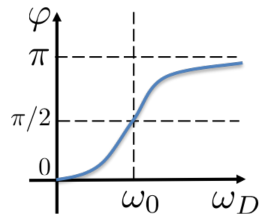
Demo: mass on a spring

Across the full parameter range of $0 < \omega_D < \infty$, the phase shift between the driving force and driven motion goes as shown in the following figure.

While the displacement of the driven motion is 90 degree out of phase with the driving force when $\omega_D = \omega_0$, the velocity of the driven motion is completely in phase with the driving force. This can be seen from

$$v(t) = \frac{dx}{dt} = \frac{F_0}{m\Gamma}\cos(\omega_D t) \tag{12}$$

which is directly proportional to the driving force. As the driving force and the velocity are always in the same direction, the driving force is always doing positive work, the system absorbs the most



energy from the driving force and goes into "resonance" and oscillates with (approximately) the largest amplitude.

Let's calculate how much energy is absorbed by the system in the driven motion for a generic value of ω_D.

$$\begin{aligned} P &= F \cdot v = F \cdot \tfrac{dx}{dt} \\ &= F_0 \cos(\omega_D t)(-Re(A)\omega_D \sin(\omega_D t) - Im(A)\omega_D \cos(\omega_D t)) \\ &= F_0\omega_D \left(-\tfrac{Re(A)}{2}\sin(2\omega_D t) - Im(A)\cos^2(\omega_D t)\right) \end{aligned} \tag{13}$$

Over time, the average power that is being absorbed is

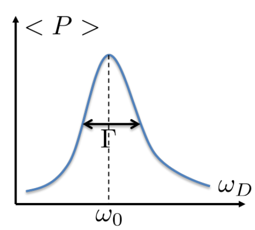
$$< P >= -\frac{1}{2}F_0\omega_D Im(A) = \frac{F_0^2\Gamma}{2m}\frac{\omega_D^2}{\omega_D^4 + (\Gamma^2 - 2\omega_0^2)\omega_D^2 + \omega_0^4} \tag{14}$$

For what value of ω_D is $< P >$ maximum? Notice that

$$< P >= \frac{F_0^2\Gamma}{2m}\frac{1}{\omega_D^2 + (\Gamma^2 - 2\omega_0^2) + \frac{\omega_0^4}{\omega_D^2}} \tag{15}$$

When $\omega_D = \omega_0$, $\omega_D^2 + \frac{\omega_0^4}{\omega_D^2}$ reaches its minimum, therefore, $< P >$ is at its max.

If we plot $< P >$ wrt to ω_D, we can see the peak of resonance, which has its maximum at $\omega_D = \omega_0$, with the maximum value being $< P >_{\text{max}}= \frac{F_0^2}{2m\Gamma}$.



How sharp is the peak? We can characterize the width of the peak using the quantity called FWHM: Full Width at Half Maximum. As the maximum is $< P >_{\text{max}}= \frac{F_0^2}{2m\Gamma}$, the height at half maximum is $< P >_{\text{half-max}}= \frac{F_0^2}{4m\Gamma}$. Correspondingly, we can find the two values of ω_D at which $< P >= \frac{F_0^2}{4m\Gamma}$ to be

$$\omega_{1/2} = \frac{1}{2}\left(\pm\Gamma + \sqrt{\Gamma^2 + 4\omega_0^2}\right) \tag{16}$$

Therefore, the FWHM is simply given by Γ.

To summarize, resonance occurs at $\omega_D = \omega_0$, with average absorbed power $< P >= \frac{F_0^2}{2m\Gamma}$ which is inversely proportional to Γ. The average absorbed power reaches its maximum because the displacement of oscillation is 90 degree out of phase with the driving force, therefore velocity of the oscillation is completely in phase with the driving force and the driving force is always doing positive work. Away from $\omega_D = \omega_0$, the absorbed power decays with FWHM Γ.

An important quantity characterizing a real oscillator is its Q factor (quality factor) which is defined as

$$Q = \frac{\omega_0}{\Gamma} \tag{17}$$

which is the ratio between the intrinsic frequency and the friction constant. Note that as ω_0 and Γ have the same dimension, Q is a dimensionless number. With higher Q, the system is going to oscillate more times before the oscillation decays if it is not driven by external forces. At the same time, if it is driven, it is going to have a narrower resonance peak, which means it is harder to tune to resonance and being more selective on frequency.

Note that if we define resonance to be when the driven oscillation has the maximum amplitude, then it happens at a slightly different location at $\omega_D = \sqrt{\omega_0^2 - \frac{\Gamma^2}{2}}$. In practice, we can often ignore this difference as long as Γ is small. We are going to look in more detail about this in the homework.

Demo 20206: Driven Spring.

Demo 20304: Driven Air Track Glider