3 Coupled Harmonic Oscillators, Normal Modes

3.2 Coupled oscillator with many DOF

Demo 20401: Three Coupled Air Track Gliders, Demo 20402: Six Coupled Pendula

Now let's look at how a coupled oscillator with many DOF oscillates. The intuitive picture is basically given by the two DOF example, but we want to be more general about the mathematical setup.

Consider a set of n EOM with n DOF involved.

$$\begin{aligned} m_1 \tfrac{d^2}{dt^2} x_1 &= F_{11} + F_{12} + ... + F_{1n} \\ &...... \\ m_n \tfrac{d^2}{dt^2} x_n &= F_{n1} + F_{n2} + ... + F_{nn} \end{aligned} \tag{1}$$

where F_{ij} is the force acting on the ith particle due to small displacement of the jth particle away from its equilibrium position.

Assume that at equilibrium, $x_1 = x_2 = ... = x_n = 0$. F_{ij} depends linearly on x_j as

$$F_{ij} = -k_{ij} x_j \tag{2}$$

Note that $k_{ij} = k_{ji}$. This is because, suppose that the total potential energy in the system which depends on x_1,..., x_n is V. Then we have

$$k_{ij} = -\frac{\partial F_i}{\partial x_j} = -\frac{\partial}{\partial x_j}\left(-\frac{\partial V}{\partial x_i}\right) = \frac{\partial^2 V}{\partial x_j \partial x_i} \tag{3}$$

At the same time, we have

$$k_{ji} = -\frac{\partial F_j}{\partial x_i} = -\frac{\partial}{\partial x_i}\left(-\frac{\partial V}{\partial x_j}\right) = \frac{\partial^2 V}{\partial x_j \partial x_i} \tag{4}$$

Therefore, $k_{ij} = k_{ji}$.

We can write the set of EOM in matrix form as

$$M \frac{d^2}{dt^2} X = -KX \tag{5}$$

where $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $M = \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_n \end{pmatrix}$, $K = \begin{pmatrix} k_{11} & \dots & k_{1n} \\ \vdots & \ddots & \vdots \\ k_{n1} & \dots & k_{nn} \end{pmatrix}$.

As long as M is invertible, we can move it to the right hand side and get

$$\frac{d^2}{dt^2} X = -M^{-1} K X \tag{6}$$

To solve this equation, we first put it into the complex form

$$\frac{d^2}{dt^2}Z = -M^{-1}KZ \tag{7}$$

and then try solutions of the form

$$Z^{(j)} = A^{(j)}e^{i\omega^{(j)}t} \tag{8}$$

$A^{(j)}$ then satisfies the eigen equation

$$M^{-1}KA^{(j)} = (\omega^{(j)})^2 A^{(j)} \tag{9}$$

Is it guaranteed that there are always solutions? If so, how do they look like? To answer this question, we make use of a math theorem:

A real symmetric $n \times n$ matrix has n real eigenvalues (two or more of them could be the same) and correspondingly n real eigenvectors which are orthogonal to each other.

In order to apply this theorem, we need to modify the eigen equation by multiplying $M^{1/2} = \begin{pmatrix} \sqrt{m_1} & & \\ & \ddots & \\ & & \sqrt{m_n} \end{pmatrix}$ from the left on both sides

$$M^{-1/2}KM^{-1/2}\left(M^{1/2}A^{(j)}\right) = (\omega^{(j)})^2\left(M^{1/2}A^{(j)}\right) \tag{10}$$

As both $M^{1/2}$ and K are symmetric real matrices, $M^{-1/2}KM^{-1/2}$ is real symmetric and there are n solutions with orthogonal eigenvectors. That is, there are n eigenmodes, characterized by eigenfrequency $\omega^{(j)}$ and eigenvector $A^{(j)}$. Note that the $A^{(j)}$'s may not be orthogonal to each other but the $M^{1/2}A^{(j)}$s are.

$$\left(A^{(k)}\right)^T MA^{(j)} = \delta_{jk} \tag{11}$$

The eigenvalue $(\omega^{(j)})^2$ is real, but it can be positive, zero or negative, corresponding to $\omega^{(j)}$ being real, zero or imaginary. What does it mean?

(1) If $(\omega^{(j)})^2 > 0$, $\omega_\pm^{(j)} = \pm\sqrt{(\omega^{(j)})^2}$ are real numbers. The general solution of the motion is

$$Z(t) = \left[a_+^{(j)}e^{i\omega_+^{(j)}t} + a_-^{(j)}e^{i\omega_-^{(j)}t}\right]A^{(j)} \tag{12}$$

Taking the real part, we get

$$X(t) = \left[\left(Re(a_+^{(j)}) + Re(a_-^{(j)})\right)\cos(\sqrt{(\omega^{(j)})^2}t) + \left(-Im(a_+^{(j)}) + Im(a_-^{(j)})\right)\sin(\sqrt{(\omega^{(j)})^2}t)\right]A^{(j)} \tag{13}$$

which describes oscillatory motion.

(2) If $(\omega^{(j)})^2 = 0$, $\omega^{(j)} = 0$. In this case, we have $Z(t) = A^{(j)}$, which is stationary. Actually, we are missing a solution $Z(t) = tV^{(j)}$, which describes constant velocity motion. So the general solution is given by

$$Z(t) = A^{(j)} + tV^{(j)} \tag{14}$$

If a system is not under the action of external force, the center of mass motion is an eigenmode with zero frequency. The center of mass of the system can remain stationary or move with a constant velocity.

(3) If $(\omega^{(j)})^2 < 0$, $\omega_\pm^{(j)} = \pm i\sqrt{-(\omega^{(j)})^2}$ are pure imaginary numbers. The general solution of the motion is

$$Z(t) = \left[a_+^{(j)} e^{-\sqrt{-(\omega^{(j)})^2}t} + a_-^{(j)} e^{\sqrt{-(\omega^{(j)})^2}t}\right] A^{(j)} \tag{15}$$

Without loss of generality, we can take a to be real. Then $X(t) = Z(t)$. The first part of the solution describes an exponentially decaying motion while the second part describes an exponentially growing motion, so that after a little while the whole motion is exponentially growing. This corresponds to motion away from an unstable equilibrium point.

Now let's focus on the case with $(\omega^{(j)})^2 > 0$ and take a more careful look at the form of the solution.

$$Z^{(j)} = a^{(j)} A^{(j)} e^{i\omega^j t} = |a^{(j)}| e^{i(\omega^{(j)}t + \varphi^{(j)})} A^{(j)} \tag{16}$$

Here $a^{(j)}$ is a complex number with phase $\varphi^{(j)}$, $A^{(j)}$ is a real vector, and $\omega^{(j)}$ is a real number.

Taking the real part of the solution, we get

$$X^{(j)} = |a^{(j)}| \cos(\omega^{(j)}t + \varphi^{(j)}) A^{(j)} = |a^{(j)}| \cos(\omega^{(j)}t + \varphi^{(j)}) \begin{pmatrix} A_1^{(j)} \\ \vdots \\ A_n^{(j)} \end{pmatrix} \tag{17}$$

That is, in an eigenmode, all the DOF oscillates with the same frequency, the same phase*, and with a fixed ratio of amplitude (note that the ratio can be negative corresponding to a π phase shift).

If all the modes are oscillatory, combining all the eigenmodes, we get the total motion

$$X(t) = \sum_j |a^{(j)}| \cos(\omega^{(j)}t + \varphi^{(j)}) A^{(j)} \tag{18}$$

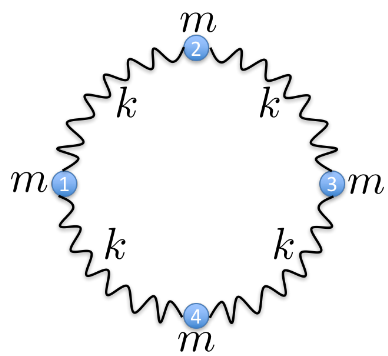
The free parameters $|a^{(j)}|$ and $\varphi^{(j)}$ are to be determined from the initial condition $x_i(0)$ and $x_i'(0)$. The n normal modes form a linearly independent (orthogonal as defined above), complete set of basis for the full motion.

4 Oscillation in an Infinite System

4.1 Translation invariant, Locally interacting system

In systems with infinite number of DOF, there are infinitely many modes, and infinitely many frequencies. We need to solve equations with infinite dimensional matrices, which becomes infinitely complicated. However, in the physical systems we are interested in, there are constraints which simplify the problem: 1. Translation symmetry 2. Local interaction. Because of this, the problem becomes solvable.

Let's start from a large but finite system with 4 particles. (Very large indeed!) The four particles are restricted to move along the ring and they are connected along the ring by springs. The whole configuration of the system is translation invariant (from 1 to 2 to 3 to 4 to 1). Moreover, each particle is only connected to its neighbors, so the interaction is local.



The EOM reads

$$\begin{aligned} m\tfrac{d^2}{dt^2}x_1 &= -k(x_1-x_2)-k(x_1-x_4) = -2kx_1+kx_2+kx_4 \\ m\tfrac{d^2}{dt^2}x_2 &= -2kx_2+kx_3+kx_1 \\ m\tfrac{d^2}{dt^2}x_3 &= -2kx_3+kx_4+kx_2 \\ m\tfrac{d^2}{dt^2}x_4 &= -2kx_4+kx_1+kx_3 \end{aligned} \tag{19}$$

In matrix form

$$\frac{d^2}{dt^2}X = -\frac{1}{m}KX \tag{20}$$

where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$, $K = \begin{pmatrix} 2k & -k & 0 & -k \\ -k & 2k & -k & 0 \\ 0 & -k & 2k & -k \\ -k & 0 & -k & 2k \end{pmatrix}$.

We can turn this into a complex equation

$$\frac{d^2}{dt^2}Z = -\frac{1}{m}KZ \tag{21}$$

and make the assumption that Z takes the form $Z = Ae^{i\omega t}$, which leads to the eigen equation

$$\frac{1}{m}KA = \omega^2 A \tag{22}$$

We can solve the eigenvalue equation and find four eigenmodes:

$$\begin{aligned} \omega^{(1)} = 0, \quad A^{(1)} &= \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \omega^{(2)} = \sqrt{\tfrac{2k}{m}}, \quad A^{(1)} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \\ \omega^{(3)} = \sqrt{\tfrac{2k}{m}}, \quad A^{(3)} &= \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad \omega^{(4)} = \sqrt{\tfrac{4k}{m}}, \quad A^{(1)} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \end{aligned} \tag{23}$$