4 Oscillation in an Infinite System

4.4 Translation invariant, locally interacting system in the Infinite Limit

Now if we increase N from 4 to ∞, we can see the continuous wave emerging. Suppose that we have a large number of mass points on the ring which are connected along the ring by springs. The equation of motion looks like

$$m\frac{d^2}{dt^2}X = -KX \tag{1}$$

where

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}, \quad K = \begin{pmatrix} 2k & -k & 0 & \dots & -k \\ -k & 2k & -k & \dots & 0 \\ 0 & -k & 2k & \dots & 0 \\ & & \ddots & & \\ -k & 0 & 0 & \dots & 2k \end{pmatrix} \tag{2}$$

Consider the complex version of the equation and assume that the solution takes the form $Z = Ae^{i\omega t}$. We get the eigen equation

$$\frac{1}{m}KA = \omega^2 A \tag{3}$$

with eigen values ω^2 and eigen vectors A. There are N independent solutions, which turn out to be

$$\left(\omega^{(j)}\right)^2 = \frac{2k}{m}(1-\cos(2\pi j/N)), \; A^{(j)} = \begin{pmatrix} e^{i2\pi j\cdot 0/N} \\ e^{i2\pi j\cdot 1/N} \\ \vdots \\ e^{i2\pi j\cdot (N-1)/N} \end{pmatrix} \tag{4}$$

The N eigen vectors can be found by solving the eigen equation of the translation operator S

$$S = \begin{pmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ & & \ddots & \\ 1 & 0 & \dots & 0 \end{pmatrix} \tag{5}$$

and we can verify that

$$SA^{(j)} = e^{i2\pi j/N}A^{(j)} \tag{6}$$

Because

$$SKS^{-1} = K \tag{7}$$

the $A^{(j)}$'s are eigen states of K as well.

A few comments about this solution:

(1) This form of $A^{(j)}$ works for any translation invariant K which satisfies

$$SKS^{-1} = K \tag{8}$$

independent of the details of the coupling!! There can be next nearest neighbor coupling, next-next-nearest-neighbor coupling, etc., the form of $A^{(j)}$ would stay the same.

I will state this in a more formal way:

For a translation invariant system of coupled harmonic oscillators described by K which satisfies $SKS^{-1} = K$, we can always find a set of eigenvectors $A^{(j)}$ s.t.

$$SA^{(j)} = e^{i2\pi j/N} A^{(j)} \tag{9}$$

That is, $A^{(j)}$ is a common set of eigenvectors for K and S with eigenvalues $\left(\omega^{(j)}\right)^2$, $e^{i2\pi j/N}$ respectively.

(2) The eigenvectors $A^{(j)}$ which are eigenvectors of both K and S represents traveling waves.

To see this, we write down the full solution and take the real part to see the full motion

$$x_n(t) = Re(z_n(t)) = Re\left(\alpha^{(j)} e^{\frac{i2\pi j}{N}(n-1)} e^{i\omega^{(j)}t}\right) = |\alpha^{(j)}| \cos\left(\omega^{(j)}t + \frac{2\pi j}{N}(n-1) + \varphi^{(j)}\right) \tag{10}$$

Now let's make some changes to the notation so that it looks more like a continuous wave. We are going to use x to label the equilibrium position of the mass particles so that

$$x = a(n-1), \quad n = 1, 2, ..., N \tag{11}$$

where a is the lattice spacing. In the limit of $N \to \infty$, $a \to 0$, x becomes continuous. To label the displacement of the wave, we are going to use ψ instead of x. Now the wave is described by

$$\psi(x,t) = |\alpha^{(j)}| \cos\left(\omega^{(j)}t + \frac{2\pi j}{L}x + \varphi^{(j)}\right) \tag{12}$$

where $L = Na$ is the total length of the system.

Now we are going to define a very important parameter for describing waves: the wave number, which unfortunately is also labeled by k.

$$k^{(j)} \equiv \frac{2\pi j}{L} \tag{13}$$

j takes value from 0 to $N-1$, so that k takes value from 0 to $\frac{2\pi(N-1)}{L}$. If we take $j = N$, i.e. $k = \frac{2\pi N}{L} = \frac{2\pi}{a}$, the form of the wave function $\psi(x,t)$ is the same as $j = 0$, i.e. $k = 0$, therefore k is periodic with period $\frac{2\pi}{a}$. Because of this periodicity, k can actually take value in any segment of length $\frac{2\pi}{a}$. A convenient choice is $[-\frac{\pi}{a}, \frac{\pi}{a})$.

Then ψ takes the form (if we omit j the mode lable)

$$\psi(x,t) = |\alpha| \cos(\omega t + kx + \varphi) \tag{14}$$

which is how waves are usually described.

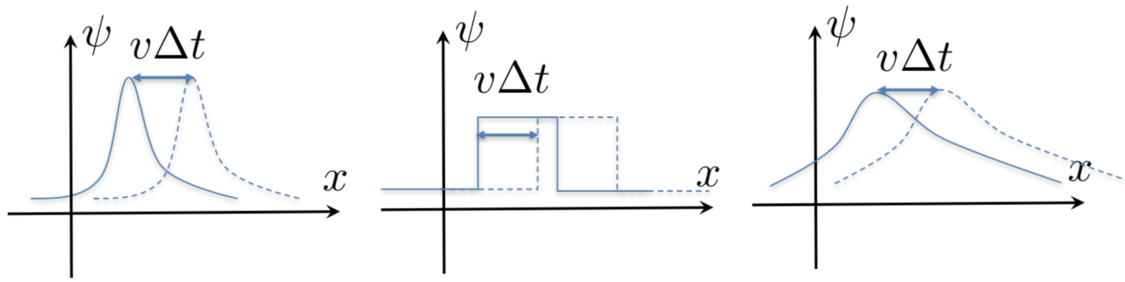
We can further write it as

$$\psi(x,t) = |\alpha| \cos\left(k(x+vt)+\varphi\right) \tag{15}$$

such that the space and time coordinates are combined in the form $x+vt$, where $v = \frac{\omega}{k}$. v is the velocity of the wave because if we shift the time coordinate by Δt and shift the spatial coordinate by $\Delta x = -v\Delta t$, the shape of the wave remains the same. In fact, this applies whenever the wave is a function of $x \pm vt$ only.

$$\psi(x,t) = f(x \pm vt) \tag{16}$$

f could take all different kinds of shape as shown in the following figure. In all cases, the wave is moving with velocity v.



The formula for the traveling wave contains some important parameters: the amplitude of the wave $|\alpha|$, the frequency ω, the wave number k and the phase φ. From them, we can find the period both in time

$$T = \frac{2\pi}{\omega} \tag{17}$$

and in space, also called the wavelength

$$\lambda = \frac{2\pi}{k} \tag{18}$$

Another important notion is the dispersion relation, which is how ω depends on k. In this example, we have

$$(\omega)^2 = \frac{2k_h}{m}\left(1-\cos(2\pi j/N)\right) = \frac{2k_h}{m}\left(1-\cos(ka)\right) \tag{19}$$

where k_h denotes the Hooke's constant for the spring while k denotes the wave number. When k is small, we have

$$(\omega)^2 \approx \frac{1}{m} k_h k^2 a^2 \tag{20}$$

so ω depends linearly on k and this is referred to as the linear dispersion relation. It is also possible to have a quadratic dispersion relation with $\omega \sim k^2$ or even higher order.

(3) The full motion, which is a superposition of all possible modes, is given by

$$\psi(x,t) = \sum_j |\alpha^{(j)}| \cos\left(\omega^{(j)}t + k^{(j)}x + \varphi^{(j)}\right) \tag{21}$$

each mode contains two free real parameters $|\alpha^{(j)}|$ and $\varphi^{(j)}$, which can be tuned to match initial conditions $\psi(x,0)$ and $\frac{\partial}{\partial t}\psi(x,t)\big|_{t=0}$.