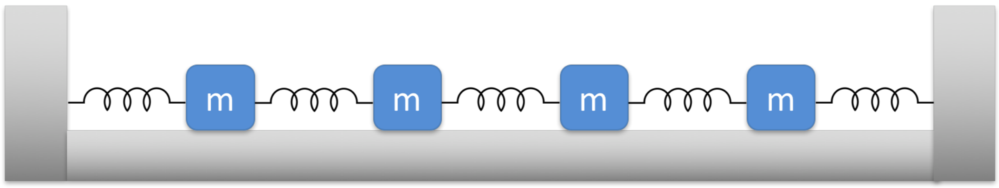
4 Oscillation in an Infinite System

4.5 Fixed Boundary condition

Consider a configuration as shown below



The system is not translation invariant any more, although in the middle part of the system it still looks translation invariant. So naively, the solution we had previously does not work. But in fact, we can use it to find the solution to the new problem.

The EOM is now given by

$$\begin{array}{l} m\frac{d^2}{dt^2}x_1 = -2k_h x_1 + k_h x_2 \\ m\frac{d^2}{dt^2}x_n = -2k_h x_n + k_h x_{n-1} + k_h x_{n+1}, \ n = 2, ..., N-1 \\ m\frac{d^2}{dt^2}x_N = -2k_h x_N + k_h x_{N-1} \end{array} \tag{1}$$

This set of EOM is still local, but not quite translation invariant.

We can extend it to a translation invariant form, by embedding the system in a larger system of size $2N+2$. The mass blocks are now labeled from $-N$ to $N+1$. This larger system is translation invariant and in order to find solutions that correspond to the original problem, we add the extra condition that $x_0 = 0$ and $x_{N+1} = 0$. That is, we imagine the two walls are also mass blocks, but they do not move.

The EOM becomes

$$\begin{array}{l} m\frac{d^2}{dt^2}x_n = -2k_h x_n + k_h x_{n-1} + k_h x_{n+1}, \ n = -N, ..., N+1 \\ x_0 = x_{N+1} = 0 \end{array} \tag{2}$$

We take periodic boundary condition so that $x_{N+2} = x_{-N}$.

This is now a translation invariant EOM which we know the solution to. But in order to get back to the original problem, we need to show: a. all solutions to Eq. 2 are solutions to Eq. 1 as well; b. all solutions to Eq. 1 can be extended to solutions to Eq. 2.

a. is easy to show because the EOM in Eq. 2 for $i = 1, ..., N$ is exactly the same as that for Eq. 1. To show b. is harder and that is what we are going to do.

Suppose that $x_1, ..., x_N$ are solutions to Eq. 1. Then $x_1, ..., x_N$ also satisfy Eq. 2 with constraint $x_0 = x_N = 0$.

Now the EOM for x_0 becomes

$$m\frac{d^2}{dt^2}x_0 = -2k_h x_0 + k_h x_1 + k_h x_{-1} \tag{3}$$

Because $x_0 = 0$, we get $x_{-1} = -x_1$. That is, we can extend the solution to x_{-1}.

Now the EOM for x_{-1} reads

$$m\frac{d^2}{dt^2}x_{-1} = -2k_h x_{-1} + k_h x_0 + k_h x_{-2} \tag{4}$$

Comparing it to the EOM for x_1 we find $x_{-2} = -x_2$.

We can continue this process and find $x_{-n} = -x_n$, $n = 1, ..., N$.

Finally we get

$$m\frac{d^2}{dt^2}x_{-N} = -2k_h x_{-N} + k_h x_{-N+1} + k_h x_{N+1} \tag{5}$$

which is consistent with the EOM for x_N and the constraint that $x_{N+1} = 0$. And the EOM for x_{N+1} reads

$$m\frac{d^2}{dt^2}x_{N+1} = -2k_h x_{N+1} + k_h x_{-N} + k_h x_N \tag{6}$$

which is consistent with $x_{N+1} = 0$.

Therefore, given a solution to Eq.1, we can consistently extend it to a solution for Eq. 2. Because of this, in order to solve Eq.1, we can solve Eq.2 instead.

We know the general solution of Eq.2, which is

$$\psi(x,t) = \sum_{j=1}^{2N+2} |\alpha^{(j)}| \cos\left(\omega^{(j)}t + k^{(j)}x + \varphi^{(j)}\right) \tag{7}$$

where $k^{(j)} = \frac{2\pi j}{(2N+2)a}$, $\left(\omega^{(j)}\right)^2 = \frac{2k_h}{m}(1 - \cos(k^{(j)}a))$.

Now impose the constraint that $x_0 = 0$ and $x_{N+1} = 0$, which corresponds to $\psi(0,t) = \psi((N+1)a, t) = 0$. This leads to the condition that

$$\begin{aligned} \psi(0,t) &= \textstyle\sum_{j=1}^{2N+2} |\alpha^{(j)}| \cos\left(\omega^{(j)}t + \varphi^{(j)}\right) = 0, \\ \psi((N+1)a,t) &= \textstyle\sum_{j=1}^{2N+2} (-)^j |\alpha^{(j)}| \cos\left(\omega^{(j)}t + \varphi^{(j)}\right) = 0 \end{aligned} \tag{8}$$

Different ω corresponds to different oscillation. To have $\psi(0,t) = \psi((N+1)a,t) = 0$ for all t, the coefficient in front of different $\cos(\omega t)$ terms must all be 0. This doesn't mean that $|\alpha^{(j)}| = 0$ because for each ω, there are two corresponding wave vectors k and $\frac{2\pi}{a} - k$. All we need to require is that these two waves cancel each other at these two points. If we take out two such modes and require them to cancel each other at $x = 0$,

$$|\alpha^{(j)}| \cos\left(\omega^{(j)}t + \varphi^{(j)}\right) + |\alpha^{(2N+2-j)}| \cos\left(\omega^{(j)}t + \varphi^{(2N+2-j)}\right) = 0 \tag{9}$$

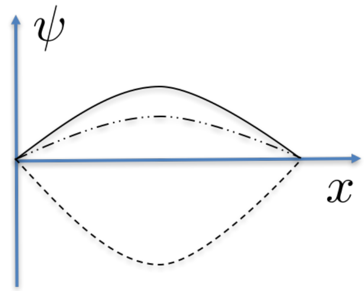
This can be achieved if $|\alpha^{(j)}| = |\alpha^{(2N+2-j)}|$ and $\varphi^{(j)} = \varphi^{(2N+2-j)} + \pi$. That is, the two oscillations have the same amplitude, the same frequency but a π phase difference and hence interfere destructively.

The form of the total wave coming from the two modes labeled by j and $2N+2-j$ is then

$$\begin{aligned} & |\alpha^{(j)}|\cos\left(\omega^{(j)}t + k^{(j)}x + \varphi^{(j)}\right) + |\alpha^{(j)}|\cos\left(\omega^{(j)}t - k^{(j)}x + \varphi^{(j)} + \pi\right) \\ = \; & |\alpha^{(j)}|\sin(\omega^{(j)}t + \varphi^{(j)})\sin(k^{(j)}x), \quad j = 1, ..., N+1 \end{aligned} \tag{10}$$

which is a standing wave.

Let's see what each of the standing wave mode looks like. For $j = 1$, $k^{(1)} = \frac{2\pi}{(2N+2)a}$, and the wave length of the standing wave (the spatial distance over which the wave pattern repeats itself) is $\lambda^{(1)} = \frac{2\pi}{k^{(1)}} = (2N+2)a = 2L$. The wavelength is twice as large as the original system! The system can only accommodate half of the wave length. If we take snap shots of the wave, it looks like



Obviously this is the largest wavelength that can satisfy the fixed boundary condition at both ends.

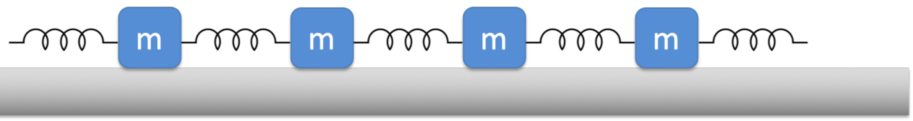
With larger j and larger k, we can accommodate more wavelength in the system. With $j = 2$, we have a full wavelength; with $j = 3$, we have one and a half of a full wave length... Finally, with $j = N+1$, we have $k^{(N)} = \frac{2\pi}{2a}$ and $\lambda = 2a$, we get a standing wave with all the N points being on the nodes so that nothing is moving. We can imagine having modes with even smaller wavelength (for example, $\lambda = a$), but this is the same as the $\lambda = 2a$ case as none of the N points move. In a standing wave, the points that do not move are called nodes; the points with the largest oscillation amplitude are called antinodes.

4.6 More Boundary Conditions

This way of solving systems with boundary conditions seems pretty complicated. But if we think about it, what we did amounts to 1. embedding the original system into a larger translation invariant system with constraints 2. make superpositions of degenerate traveling wave modes of the larger system so that the new mode satisfy the constraint 3. reduce the solution back to the original system. To make this process work, we need to determine: 1. what is the size of the larger translation invariant system 2. how to make superposition of the traveling wave modes such that it satisfies the constraints. The key to these questions is to figure out the wave length of the traveling mode that fits with the boundary condition. Once the wave length is determined, we make superposition of the two degenerate traveling waves with the same frequency but traveling in opposite directions so that the constraints are satisfied.

4.6.1 Open boundary condition

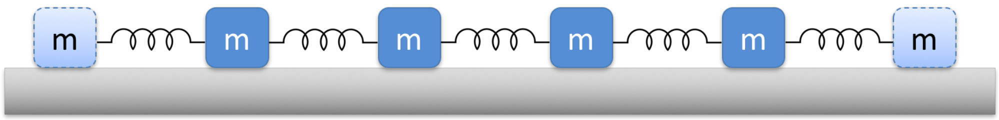
Let's try to use this method to find the eigenmodes in a system with open boundary conditions. Consider the following coupled harmonic oscillator system with free ends.



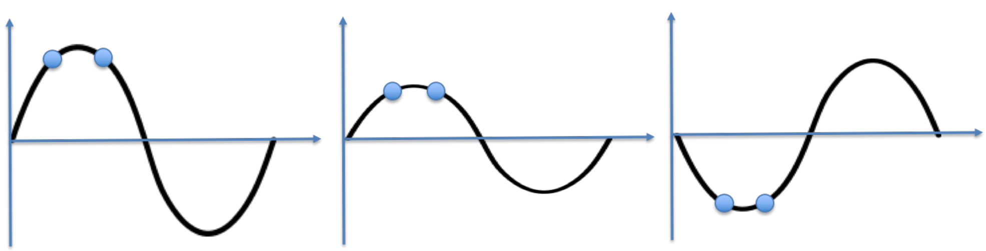
If we try to embed it into a larger translation invariant system, what constraint does the translation invariant system have to satisfy? If we label the mass blocks in the original system as 1 through N, then the constrain is such that

$$x_0 = x_1, \quad x_N = x_{N+1} \tag{11}$$

because we can imagine there are virtual mass blocks 0 and $N+1$, but as there is no force between 0 and 1, N and $N+1$, the spring between them is neither compressed nor stretch and therefore 0 and 1, N and $N+1$ have to have the same displacement.



It is not possible to have two neighboring points have the same displacement at all time in a traveling wave, but it is possible in a standing wave. In a standing wave, if the two neighboring points are on the two sides of an antinote – point with maximum oscillation amplitude (with equal distance), they have the same displacement.



Therefore, in order to satisfy the constraint, we need to have a standing wave, with oscillation maximum between 0 and 1 and between N and $N+1$. What are the wavelengths that can possibly satisfy this? The total length between these two antinodes is Na. We can fit as little as half a wavelength between them, which corresponds to a wavelength of $2Na$.

More generally, the allowed wavelengths are

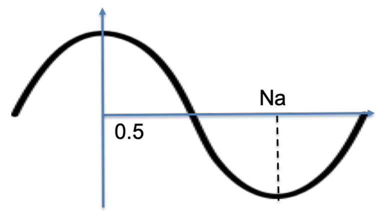
$$2Na, Na, \frac{2}{3}Na, ..., \frac{2}{n}Na, ... \tag{12}$$

Having fixed the wavelength $\lambda_n = \frac{2}{n}Na$, we take the corresponding two traveling wave modes and make superpositions of them such that they form the desired standing wave mode.

$$\psi^{(n)}(x,t) = |\alpha^n| \cos\left(\omega^{(n)}t + k^{(n)}x + \varphi^{(n)}\right) + |\alpha^{(-n)}| \cos\left(\omega^{(n)}t - k^{(n)}x + \varphi^{(-n)}\right) \tag{13}$$

$|\alpha^n|$, $|\alpha^{(-n)}|$, $\varphi^{(n)}$ and $\varphi^{(-n)}$ need to be chosen such that $\frac{d}{dx}\psi(0,t) = \frac{d}{dx}\psi(Na,t) = 0$. It follows that $|\alpha^n| = |\alpha^{(-n)}|$, $\varphi^{(n)} = \varphi^{(-n)}$. Therefore,

$$\psi^{(n)}(x,t) = 2|\alpha^n| \cos\left(\omega^{(n)}t + \varphi^{(n)}\right) \cos\left(k^{(n)}x\right) \tag{14}$$



4.6.2 Driven boundary condition

We are going to deal with this case in the homework.

Demo 20608: Rubens' Tube

Demos: Resonance and Normal Modes