4 Oscillation in an Infinite System

4.7 Wave equation in the continuum limit

In the previous discussion, we started from a set of discrete equation of motion, found the solution and then took the continuum limit for the solution. What is the continuum version of the equation of motion?

In the discrete case, the equation of motion for a particular x is given by

$$m\frac{\partial^2}{\partial t^2}\psi(x,t) = -2k_h\psi(x,t)+k_h\psi(x-a,t)+k_h\psi(x+a,t) = k_h a\left[\frac{\psi(x+a,t)-\psi(x,t)}{a} - \frac{\psi(x,t)-\psi(x-a,t)}{a}\right] \tag{1}$$

In the limit of $a \to 0$, we can replace discrete subtraction with differentiation. That is, $\frac{\psi(x+a,t)-\psi(x,t)}{a}$ can be replaced by $\frac{\partial}{\partial x}\big|_x$, $\frac{\psi(x,t)-\psi(x-a,t)}{a}$ can be replaced by $\frac{\partial}{\partial x}\big|_{x-a}$. The right hand side then becomes a second order derivative

$$k_h a\left[\frac{\psi(x+a,t)-\psi(x,t)}{a} - \frac{\psi(x,t)-\psi(x-a,t)}{a}\right] \to k_h a^2\frac{\partial^2}{\partial x^2}\psi(x,t) \tag{2}$$

The EOM relates the second order derivative in time to the second order derivative in space

$$m\frac{\partial^2}{\partial t^2}\psi = k_h a^2\frac{\partial^2}{\partial x^2}\psi \tag{3}$$

m and $k_h a^2$ are not good parameters for describing a continuous system. Instead we define

$$\rho \equiv \frac{m}{a},\ \ \mu \equiv k_h a \tag{4}$$

which represents respectively mass per unit length and spring constant per unit length (why?).

The EOM can then be written as

$$\rho\frac{\partial^2}{\partial t^2}\psi = \mu\frac{\partial^2}{\partial x^2}\psi \tag{5}$$

The eigenmodes are then given by

$$\psi(x,t) = |\alpha|\cos(\omega t - kx + \varphi) \tag{6}$$

where $-\frac{\pi}{a} \le k < \frac{\pi}{a}$. As $a \to 0$, the range of k is basically infinite. The relation between ω and k can be found from the EOM to be

$$\rho\omega^2 = \mu k^2 \tag{7}$$

so that

$$\omega = \sqrt{\frac{\mu}{\rho}}|k| \tag{8}$$

This is exactly the linear dispersion relation we found previously in the small k limit. In the continuum formulation, the whole dispersion relation becomes linear, which implies that the continuum formulation only describes waves with wave length much longer than the atomic scale. For vibration on the atomic scale, this continuum description breaks down.

The velocity of the each mode is

$$v = \frac{\omega}{|k|} = \sqrt{\frac{\mu}{\rho}} \tag{9}$$

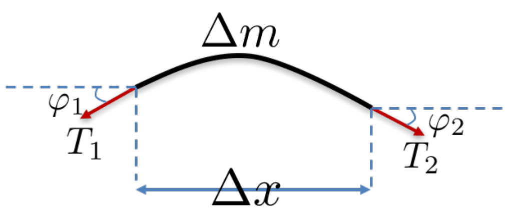
which is the same for all modes (in the long wavelength limit).

5 Various types of waves

Not every system can be described as mass blocks coupled by springs. For different types of waves, we need to look into the physical mechanism generating the wave to establish the EOM.

5.1 Transverse wave on a string

Consider the vibrational motion of a string in the transverse direction to its length. To establish the EOM, we break the string down to small segments as shown in the figure.



The small segment is being pulled by the segments on its two sides in the direction tangent to the string. The horizontal part of the two forces cancel each other

$$T_1 \cos(\varphi_1) = T_2 \cos(\varphi_2) = T \tag{10}$$

The vertical part adds up

$$-T_1 \sin(\varphi_1) - T_2 \sin \varphi_2 = -T(\tan(\varphi_1) + \tan(\varphi_2)) \tag{11}$$

On the other hand,

$$\tan(\varphi_1) = \frac{\partial \psi}{\partial x}\Big|_x, \ \tan(\varphi_2) = -\frac{\partial \psi}{\partial x}\Big|_{x+\Delta x} \tag{12}$$

Therefore we get

$$-T(\tan(\varphi_1) + \tan(\varphi_2)) = -T(-\frac{\partial^2 \psi}{\partial x^2})\Delta x \tag{13}$$

And according to Newton's law

$$-T(-\frac{\partial^2 \psi}{\partial x^2})\Delta x = \Delta m \frac{\partial^2 \psi}{\partial t^2} \tag{14}$$

So we find the EOM to be

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 \psi}{\partial x^2} \tag{15}$$

where ρ is the mass density of the string.

This EOM takes exactly the same form as that in the mass coupled to spring example if we replace $\frac{\mu}{\rho}$ by $\frac{T}{\rho}$. The traveling wave modes are given by

$$\psi(x,t) = |\alpha| \cos(\omega t - kx + \varphi) \tag{16}$$

and the dispersion relation is given by

$$\omega = \sqrt{\frac{T}{\rho}} |k| \tag{17}$$

The velocity of the wave in a string is hence inversely proportional to $\sqrt{\rho}$ and proportional to $\sqrt{T}$.

If we enforce fixed boundary condition on a string of length L, we know that the possible modes are standing waves with wavelength $\lambda^{(n)} = \frac{2L}{n}$, where n is a positive integer. In each mode, the standing wave is described by

$$\psi^{(n)}(x,t) = |\alpha^{(n)}| \sin(k^{(n)} x) \cos(\omega^{(n)} t + \varphi^{(n)}) \tag{18}$$

where $k^{(n)} = \frac{2\pi}{\lambda^{(n)}} = \frac{n\pi}{L}$, $\omega^{(n)} = vk^{(n)}$ while $|\alpha^{(n)}|$ and $\varphi^{(n)}$ are free parameters. The general form of motion satisfying this boundary condition is then

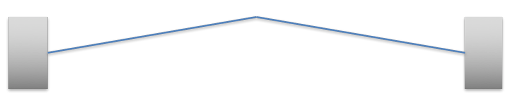
$$\psi(x,t) = \sum_n |\alpha^{(n)}| \sin(k^{(n)} x) \cos(\omega^{(n)} t + \varphi^{(n)}) \tag{19}$$

The longest wave length that can fit into the system is $2L$. The base frequency of the string is then given by

$$\omega = \sqrt{\frac{T}{\rho}} \frac{2\pi}{2L} = \sqrt{\frac{T}{\rho}} \frac{\pi}{L} \tag{20}$$

so the tune made by the string will be higher if the tension T is higher, or if the density ρ is lower, or if the total length L is shorter.

Let's now put in an initial condition for the string, for example, a string plucked as shown in this figure.



Suppose the initial displacement and initial velocity are given by $\psi(x,0) = h(x)$, $\frac{\partial}{\partial t}\psi(x,0) = v(x)$. The free parameters $|\alpha^{(n)}|$ and $\varphi^{(n)}$ can be determined from

$$\begin{aligned} h(x) &= \textstyle\sum_n |\alpha^{(n)}| \sin(k^{(n)} x) \cos(\varphi^{(n)}) \\ v(x) &= \textstyle\sum_n -\omega^{(n)} |\alpha^{(n)}| \sin(k^{(n)} x) \sin(\varphi^{(n)}) \end{aligned} \tag{21}$$

$h(x)$ is not necessarily periodic between 0 and L but it can be extended into a periodic function with periodic $2L$. With that extension, the above expression becomes the Fourier expansion of $h(x)$ and $v(x)$ and the coefficient of the expansion can be derived from

$$\begin{aligned} |\alpha^{(n)}| \cos(\varphi^{(n)}) &= \tfrac{2}{L} \textstyle\int dx h(x) \sin(k^{(n)} x) \\ -\omega^{(n)} |\alpha^{(n)}| \sin(\varphi^{(n)}) &= \tfrac{2}{L} \textstyle\int dx v(x) \sin(k^{(n)} x) \end{aligned} \tag{22}$$

5.2 Fourier Analysis

In the last step of the derivation, we have used the Fourier transform.

Let's recall the Fourier transform for real periodic functions:

Any reasonably continuous periodic function with period L can be expressed as an infinite sum of sines and cosines

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos\left(\frac{2\pi n x}{L}\right) + b_n \sin\left(\frac{2\pi n x}{L}\right) \right) \tag{23}$$

As the sin and cos functions are orthonormal in the following way

$$\begin{aligned} \int_0^L dx \sin\left(\tfrac{2\pi n x}{L}\right) \sin\left(\tfrac{2\pi n' x}{L}\right) &= \tfrac{L}{2}\delta_{nn'} \\ \int_0^L dx \cos\left(\tfrac{2\pi n x}{L}\right) \cos\left(\tfrac{2\pi n' x}{L}\right) &= \tfrac{L}{2}\delta_{nn'} \\ \int_0^L dx \sin\left(\tfrac{2\pi n x}{L}\right) \cos\left(\tfrac{2\pi n' x}{L}\right) &= 0 \end{aligned} \tag{24}$$

The coefficients a_n and b_n can be found from

$$\begin{aligned} a_n &= \tfrac{2}{L}\int_0^L dx f(x) \cos\left(\tfrac{2\pi n x}{L}\right) \\ b_n &= \tfrac{2}{L}\int_0^L dx f(x) \sin\left(\tfrac{2\pi n x}{L}\right) \end{aligned} \tag{25}$$

If $f(x)$ is a complex function, a very similar expansion hold

$$f(x) = \sum_{n=-N}^{N} C_n e^{ik_n x}, \quad k_n = \frac{2\pi n}{L} \tag{26}$$

If the function $f(x)$ is not necessarily periodic, the discrete sum becomes a continuous integral

$$\begin{aligned} f(x) &= \int_{-\infty}^{+\infty} \tilde{f}(k) e^{ikx} dk \\ \tilde{f}(k) &= \tfrac{1}{2\pi}\int_{-\infty}^{+\infty} f(x) e^{-ikx} dx \end{aligned} \tag{27}$$

These two formulas are called the Fourier transform and the inverse Fourier transform respectively.

Some useful Fourier transform identities:

$$\begin{array}{ll} f(x) = 1 & \tilde{f}(k) = \delta(k) \\ f(x) = \delta(x) & \tilde{f}(k) = \frac{1}{2\pi} \\ f(x) = e^{iax} & \tilde{f}(k) = \delta(k-a) \\ f(x) = \cos(ax) & \tilde{f}(k) = \frac{1}{2}\left(\delta(k-a) + \delta(k+a)\right) \\ f(x) = \sin(ax) & \tilde{f}(k) = -\frac{i}{2}\left(\delta(k-a) - \delta(k+a)\right) \end{array} \tag{28}$$