5 Various types of waves

5.4 Light wave in free space

The electromagnetic wave is fundamentally different from the other types of waves that we have talked about. For sound wave or water wave, wave is the periodic oscillatory motion of particles in some underlying medium. For the electromagnetic wave in the LC circuit example, we can also identify wave as the periodic motion of the electrons in the circuit. But once we remove the circuit elements and have electromagnetic wave propagate in vacuum, it does not correspond to any real motion any more. Instead, what is oscillating is the electric and magnetic field.

Light wave in free space is described by Maxwell's equation. To understand what Maxwell equation is saying, first we need to realize that in free space light wave can propagate in all possible directions, so spatial dimension increases from one (x) to three (x, y, z). Moreover, electric field and magnetic field are vectors and have three components as well (E_x, E_y, E_z, B_x, B_y, B_z). Using vector notation, Maxwell's equation reads

$$\begin{cases} \nabla \cdot \vec{E} = 0 \\ \nabla \cdot \vec{B} = 0 \\ \frac{\partial}{\partial t}\vec{B} = -\nabla \times \vec{E} \\ \frac{\partial}{\partial t}\vec{E} = \frac{1}{\mu_0 \epsilon_0}\nabla \times \vec{B} \end{cases} \tag{1}$$

∇ is the vector differential operator $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$.

If we write down each component equation explicitly, we get

$$\begin{cases} \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0, \\ \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0, \\ \frac{\partial B_x}{\partial t} = \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y}, \quad \frac{\partial B_y}{\partial t} = \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}, \quad \frac{\partial B_z}{\partial t} = \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} \\ \frac{\partial E_x}{\partial t} = \frac{1}{\mu_0\epsilon_0}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right), \quad \frac{\partial E_y}{\partial t} = \frac{1}{\mu_0\epsilon_0}\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right), \quad \frac{\partial E_z}{\partial t} = \frac{1}{\mu_0\epsilon_0}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) \end{cases} \tag{2}$$

This does not look like any of the wave equation we have seen before because this is a set of first order differential equation while all the wave equations we have seen so far involve second order differentiation. Actually, the Maxwell equations can be turned into the familiar form if we take time derivative on the two sides of the third equation and get

$$\frac{\partial^2}{\partial t^2}\vec{B} = -\nabla \times \left(\frac{\partial}{\partial t}\vec{E}\right) = -\frac{1}{\mu_0\epsilon_0}\nabla \times \nabla \times \vec{B} \tag{3}$$

Using the first equation $\nabla \cdot \vec{B} = 0$, this simplifies into

$$\frac{\partial^2}{\partial t^2}\vec{B} = -\frac{1}{\mu_0\epsilon_0}\left(\nabla(\nabla \cdot \vec{B}) - \nabla^2\vec{B}\right) = \frac{1}{\mu_0\epsilon_0}\nabla^2\vec{B} \tag{4}$$

Similarly, we can get

$$\frac{\partial^2}{\partial t^2}\vec{E} = -\frac{1}{\mu_0\epsilon_0}\left(\nabla(\nabla \cdot \vec{E}) - \nabla^2\vec{E}\right) = \frac{1}{\mu_0\epsilon_0}\nabla^2\vec{E} \tag{5}$$

If we take one of the components, we can see that it looks just like the wave equation we have seen before, only with three spatial coordinates. For example, we have

$$\frac{\partial^2}{\partial t^2}B_x = \frac{1}{\mu_0\epsilon_0}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)B_x \tag{6}$$

In particular, we can read directly from the equation that the light velocity is $v = \frac{1}{\sqrt{\mu_0\epsilon_0}}$, which is 3×10^8m/s, the speed of light.

In these second order wave equations, different components of electric and magnetic fields are decoupled from each other. But in Maxwell equation, they are coupled. Let's consider the simpler case where the wave does not depend on spatial coordinate x and y but only on z and see how the components are coupled to each other. The set of Maxwell equations simplify into

$$\begin{cases} \frac{\partial}{\partial z}E_z = 0, \quad \frac{\partial}{\partial z}B_z = 0 \\ \frac{\partial}{\partial t}E_z = 0, \quad \frac{\partial}{\partial t}B_z = 0 \\ \frac{\partial}{\partial t}E_x = -\frac{1}{\mu_0\epsilon_0}\frac{\partial B_y}{\partial z}, \quad \frac{\partial}{\partial t}E_y = \frac{1}{\mu_0\epsilon_0}\frac{\partial B_x}{\partial z} \\ \frac{\partial}{\partial t}B_x = \frac{\partial E_y}{\partial z}, \quad \frac{\partial}{\partial t}B_y = -\frac{\partial E_x}{\partial z} \end{cases} \tag{7}$$

As E_z and B_z has no spatial or time variation, to avoid infinite electromagnetic energy, they must be zero in the whole space. Only the E_x, E_y and B_x and B_y components are nonzero; the light wave is said to be transverse.

Due to translation symmetry, we expect the solution to be

$$E_x = \mathcal{E}_x e^{i(kz-\omega t)}, E_y = \mathcal{E}_y e^{i(kz-\omega t)}, B_x = \mathcal{B}_x e^{i(kz-\omega t)}, B_y = \mathcal{B}_y e^{i(kz-\omega t)}, \tag{8}$$

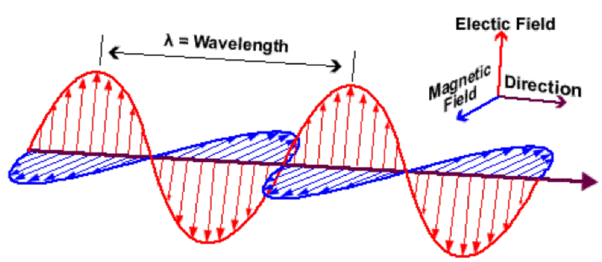
Here the $\mathcal{E}_x$, $\mathcal{E}_y$, $\mathcal{B}_x$, $\mathcal{B}_y$ are complex numbers characterizing both the amplitude and phase factor of the wave and they are related by

$$-k\mathcal{E}_y = \omega\mathcal{B}_x, k\mathcal{E}_x = \omega\mathcal{B}_y, -k\mathcal{B}_y = -\mu_0\epsilon_0\omega\mathcal{E}_x, k\mathcal{B}_x = -\mu_0\epsilon_0\omega\mathcal{E}_y, \tag{9}$$

If we define $\vec{k}$ as the vector with amplitude k and pointing in the direction of z, we have $\vec{k} \perp \vec{E}$, $\vec{k} \perp \vec{B}$, $\vec{E} \perp \vec{B}$ and they are related by

$$\vec{B} = \frac{1}{\omega}\vec{k} \times \vec{E} \tag{10}$$

If we take a snapshot of the light wave, it looks like



This is called a plane wave because if at a particular time, we find all the points with the same electric field vector, such points would form parallel planes perpendicular to the z axis. Plane waves have a single frequency, a single wave number (wave length) and is the simplest form and basic building block of more complicated wave forms. For plane wave propagating in the z direction, the E field can point in any direction in the $x - y$ plane. The direction the E field points to is called the polarization of the plane wave.

5.5 Heat wave

Not all waves satisfy the same wave equation. The heat wave is one example which does not.

Consider a metal rod (of small radius) with the left end connected to a heat bath of temperature T_1, the right end connected to a heat bath of temperature T_2. At $t = 0$, the rod has some temperature profile $T_0(x)$, $(T_0(0) = T_1, T_0(L) = T_2)$. How does the temperature along the rod change with time?

First, let's establish the equation of motion. Consider a small segment in the rod of length Δx. Heat entering from the left end during time Δt is

$$q_L = -\mu \frac{\partial T}{\partial x}\bigg|_x \Delta t \tag{11}$$

Heat leaving from the right end is

$$q_R = -\mu \frac{\partial T}{\partial x}\bigg|_{x+\Delta x} \Delta t \tag{12}$$

The total amount of heat going into this segment is

$$q = q_L - q_R = \mu \frac{\partial^2 T}{\partial x^2} \Delta x \Delta t \tag{13}$$

Correspondingly, the change in temperature of this segment is

$$\Delta T = \frac{q}{\Delta x \rho A c} \tag{14}$$

where c is the specific heat per unit mass, A is the area of the cross section of the rod.

The wave equation is hence given by

$$\frac{\Delta T}{\Delta t} = \frac{\mu}{\rho A c} \frac{\partial^2 T}{\partial x^2} \tag{15}$$

In the continuum limit, the equation becomes

$$\frac{\partial T}{\partial t} = \frac{\mu}{\rho A c} \frac{\partial^2 T}{\partial x^2} \tag{16}$$

If we try to plug a traveling wave $T = T_0 e^{i(\omega t - kx)}$ into the equation, we find that

$$i\omega = -\frac{\mu}{\rho A c} k^2 \tag{17}$$

therefore, k and ω cannot both be real. Complex k and complex ω usually corresponds to decaying wave either with space or with time in physically reasonable situations. This makes sense because if we initiate the system with some temperature profile, we expect the gradient to disappear and the medium to achieve thermal equilibrium in the end.