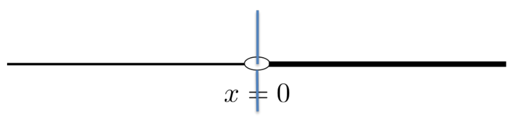
6 Reflection and Transmission

6.2 Transmission and reflection

6.2.2 Example 2

Now let's consider a second example where the two string segments are connected in a different way. The two strings are connected to a massless ring which is constrained to move vertically along a rod.



Compared with the previous case, one difference is that the tension in the two string segments can be different, depending on how much they are tightened, so it represents a more generic type of boundary condition. Suppose the tensions are T_L and T_R respectively. If we send in a wave from the left, what is the reflected and transmitted wave?

The total wave at $x < 0$ and $x > 0$ are still respectively

$$\begin{cases} \psi_L(x,t) = A_i e^{i(\omega t - k_L x)} + A_r e^{i(\omega t + k_L x)} \\ \psi_R(x,t) = A_t e^{i(\omega t - k_R x)} \end{cases} \tag{1}$$

At $x = 0$, we still have continuity of the string which means

$$\psi_L(0,t) = \psi_R(0,t) \tag{2}$$

At the same time, the vertical part of the force should balance which leads to the condition

$$T_L \frac{\partial \psi_L}{\partial x}(0,t) = T_R \frac{\partial \psi_R}{\partial x}(0,t) \tag{3}$$

From these two conditions we find

$$\frac{A_t}{A_i} = \frac{2T_L k_L}{T_L k_L + T_R k_R}, \quad \frac{A_r}{A_i} = \frac{T_L k_L - T_R k_R}{T_L k_L + T_R k_R} \tag{4}$$

When $T_L k_L = T_R k_R$, there is no reflection. That is

$$\frac{T_L}{v_L} = \frac{T_R}{v_R} \tag{5}$$

As $v = \sqrt{\frac{T}{\rho}}$, this condition is equivalent to

$$\sqrt{T_L \rho_L} = \sqrt{T_R \rho_R} \tag{6}$$

or in other words

$$z_L = z_R \tag{7}$$

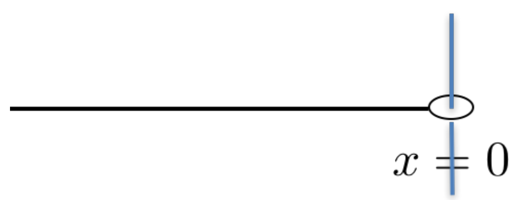
So we have arrived at the same conclusion as in the previous case: there is no reflection when the impedance on the two sides matches.

We can rewrite the relation between the amplitudes as

$$T = \frac{A_t}{A_i} = \frac{2z_L}{z_L + z_R}, \ \ R = \frac{A_r}{A_i} = \frac{z_L - z_R}{z_L + z_R} \tag{8}$$

which is of exactly the same form as in the previous example. When $z_L > z_R$, A_r is in phase with A_i; when $z_L < z_R$, A_r is 180 degree out of phase with A_i.

6.2.3 Example 3



Now let's consider a third example where a semi infinite string terminates at a massless ring that is constrained to move vertically with friction constant γ. Nothing oscillates on the right hand side of $x = 0$. But as we are going to see, to an observer on the left hand side of $x = 0$, it is as if there were a string on the right hand side with impedance γ.

At $x = 0$, because the ring is massless, the total force acting on it should be zero. In particular, in the vertical direction, the force from the string is balanced by the friction from the rod

$$T\frac{\partial \psi}{\partial x}\Big|_{x=0} = -\gamma \frac{\partial \psi}{\partial t}\Big|_{x=0} \tag{9}$$

The total wave on the left hand side of $x = 0$ is

$$\psi_L(x,t) = A_i e^{i(\omega t - k_L x)} + A_r e^{i(\omega t + k_L x)} \tag{10}$$

The boundary condition at $x = 0$ tells us

$$Tk_L(A_r - A_i) = -\gamma\omega(A_i + A_r) \tag{11}$$

Therefore,

$$R = \frac{A_r}{A_i} = \frac{Tk_L - \gamma\omega}{Tk_L + \gamma\omega} \tag{12}$$

In terms of reflection, a massless ring with friction is equivalent to a semi-infinite string with

$$T_R k_R = \gamma\omega \ \Rightarrow \ \gamma = \frac{T_R k_R}{\omega} = \frac{T_R}{v_R} = \sqrt{T_R \rho_R} = z_R \tag{13}$$

In fact, if we directly compute the impedance of the ring using the definition that impedance is the ratio between force and velocity, we find

$$z_R = \left(\gamma \frac{\partial \psi}{\partial t}\right) \Big/ \frac{\partial \psi}{\partial t} = \gamma \tag{14}$$

Therefore, this example shows that two systems generate the same reflection wave if they have the same impedance, even though physically they look totally different. This has to be the case because the interaction in the system is local. The only thing the left hand side knows about the right hand side is how much motion is generated ($\frac{\partial \psi}{\partial t}$) by a certain force ($T\frac{\partial \psi}{\partial x}$). Therefore, two systems with the same impedance look the same to the left hand side of the system.

Question: what if we send in a pulse, instead of a traveling wave? What is the relation between the reflected pulse and the incident pulse?

6.2.4 Example 4

Consider a similar setup as in the previous example but with a massive ring (m) and no friction. In this case, the force and velocity of the ring does not have a simple linear relation. Instead we have

$$-T\frac{\partial \psi}{\partial x}\Big|_{x=0} = m\frac{\partial^2 \psi}{\partial t^2}\Big|_{x=0} \tag{15}$$

How does the reflected wave change? The total wave on the left hand side is again

$$\psi_L(x,t) = A_i e^{i(\omega t - k_L x)} + A_r e^{i(\omega t + k_L x)} \tag{16}$$

The boundary condition at $x = 0$ tells us

$$iTk_L(A_i - A_r) = -m\omega^2(A_i + A_r) \tag{17}$$

Therefore,

$$R = \frac{A_r}{A_i} = \frac{iTk_L + m\omega^2}{iTk_L - m\omega^2} \tag{18}$$

which is not a real number!

This simply means that the reflected wave is phase shifted relative to the input wave. The effective impedance of the ring is

$$z_R = im\omega \tag{19}$$

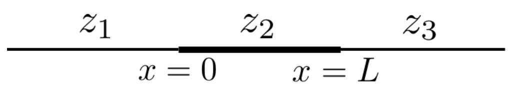
which is 1. pure imaginary 2. ω dependent. In general, impedance can be complex and the amplitude of reflection is always related to the amplitude of incoming wave as

$$\frac{A_r}{A_i} = \frac{z_L - z_R}{z_L + z_R} \tag{20}$$

What does a purely imaginary impedance mean for the power transfer in the system? When the impedance is purely imaginary, the velocity of the ring and the force acting on it has a $\pi/2$ phase shift. Therefore, on average, the ring does not absorb any power. All the power that comes from the input wave gets reflected back.

6.3 Multiple Reflections

Consider the situation where three string segments are connected in series. Both the string density and tension can change from one segment to another. If a wave is sent in from the left as $A_i e^{i(\omega t - k_1 x)}$,



part of it will get reflected at $x = 0$, part of it will get transmitted to $x = L$. The transmitted part will be partly reflected at $x = L$ and partly transmitted. The reflection at $x = L$ will be partly reflected at $x = 0$ again and partly transmitted, and so on and so forth...

This sounds like a complicated process, but what is the total wave that gets reflected or transmitted? As we are going to see, to find that the calculation is not too complicated and the result is highly useful because we will learn how to choose the segments to minimize or maximize transmission / reflection.

Let's write the wave in the left most segment as

$$\psi_1(x,t) = A_i e^{i(\omega t - k_1 x)} + A_r e^{i(\omega t + k_1 x)} \tag{21}$$

The wave between $x = 0$ and $x = L$ can be written as

$$\psi_2(x,t) = A_L e^{i(\omega t + k_2 x)} + A_R e^{i(\omega t - k_2 x)} \tag{22}$$

The wave to the right of $x = L$ can be written as

$$\psi_3(x,t) = A_t e^{i(\omega t - k_3 x)} \tag{23}$$

From the boundary condition, we can determine the relation between A_i, A_r and A_t. The continuity of the wave at $x = 0$ and $x = L$ gives

$$\begin{cases} A_i + A_r = A_L + A_R \\ A_L e^{ik_2 L} + A_R e^{-ik_2 L} = A_t e^{-ik_3 L} \end{cases} \tag{24}$$

Another boundary condition comes from the balance of force at $x = 0$ and $x = L$. In particular, the vertical component of the forces coming from the strings should balance each other. We know that the vertical component of the force is given by

$$F = z\frac{\partial \psi}{\partial t} \tag{25}$$

Note that if the traveling wave generated is traveling to the left, we need a force that is equal in amplitude but opposite in direction.

$$F = -z\frac{\partial \psi}{\partial t} \tag{26}$$

With this, the balance of force at $x = 0$ and $x = L$ gives

$$\begin{cases} z_1(A_i - A_r) = z_2(A_R - A_L) \\ z_2(A_R e^{-ik_2 L} - A_L e^{ik_2 L}) = z_3 A_t e^{-ik_3 L} \end{cases} \tag{27}$$

Combining these equations we find, after some algebra

$$\frac{A_r}{A_i} = \left(z_2 + z_1 - \frac{z_2 + z_3}{z_2 - z_3}(z_2 - z_1)e^{i2k_2 L}\right) \Big/ \left(\frac{z_2 + z_3}{z_2 - z_3}e^{i2k_2 L}(z_2 + z_1) - (z_2 - z_1)\right) \tag{28}$$

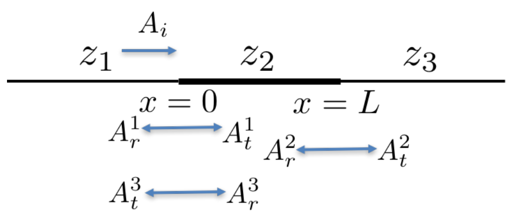
This looks very complicated but let's look at the special cases where $z_3 = z_1$, that is, the wave passes through some intermediate medium (a think film or prism) in an otherwise uniform medium. Let's also restrict our attention to only the case where z_1 and z_2 are real. Then we get

$$\frac{A_r}{A_i} = \Big((z_2 + z_2)(z_2 - z_1)(1 - e^{i2k_2L})\Big) \Big/ \Big((z_2 + z_1)^2 e^{i2k_2L} - (z_2 - z_1)^2\Big) \tag{29}$$

Therefore, whenever $2k_2L = 2\pi n$ (n is integer), $A_r = 0$. That is, the incoming wave does not get reflected at all when the length of the intermediate region is

$$L = \frac{n\pi}{k_2} = \frac{1}{2}n\lambda_2 \tag{30}$$

How to understand this? Consider the sequential reflection process.



A_t^3 and A_r^1 are both part of the total reflection wave. If they have the same phase, they interfere constructively with each other, giving rise to a large reflected wave; if they are out of phase with each other, they interfere destructively and give rise to a small reflection wave. So what is the relation between their phases? The phase difference between A_r^1 and A_t^3 comes from the following sources:

(1) phase difference between A_r^1 and A_t^1.

(2) propagation of the wave form $x = 0$ to $x = L$.

(3) phase difference between A_t^1 and A_r^2.

(4) propagation of the wave from $x = L$ back to $x = 0$.

When z_1 and z_2 are real, (1) and (3) add up to -1 while (2) and (4) add up to e^{i2k_2L}. Therefore, whenever $e^{i2k_2L} = 1$, i.e. $L = \frac{n\pi}{k_2}$, A_t^3 and A_r^1 are out of phase and interference destructively.