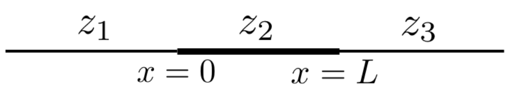
6 Reflection and Transmission

6.3 Multiple Reflections



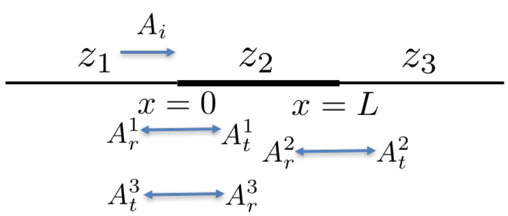
Last time we considered a multiple reflection situation and found that when z_1 and z_3 are equal

$$\frac{A_r}{A_i} = \Big((z_2 + z_2)(z_2 - z_1)(1 - e^{i2k_2L})\Big) \Big/ \Big((z_2 + z_1)^2 e^{i2k_2L} - (z_2 - z_1)^2\Big) \tag{1}$$

Therefore, whenever $2k_2L = 2\pi n$ (n is integer), $A_r = 0$. That is, the incoming wave does not get reflected at all when the length of the intermediate region is

$$L = \frac{n\pi}{k_2} = \frac{1}{2}n\lambda_2 \tag{2}$$

How to understand this? Consider the sequential reflection process.



A_t^3 and A_r^1 are both part of the total reflection wave. If they have the same phase, they interfere constructively with each other, giving rise to a large reflected wave; if they are out of phase with each other, they interfere destructively and give rise to a small reflected wave. So what is the relation between their phases? The phase difference between A_r^1 and A_t^3 comes from the following sources:

(1) phase difference between A_r^1 and A_t^1.

(2) propagation of the wave form $x = 0$ to $x = L$.

(3) phase difference between A_t^1 and A_r^2.

(4) propagation of the wave from $x = L$ back to $x = 0$.

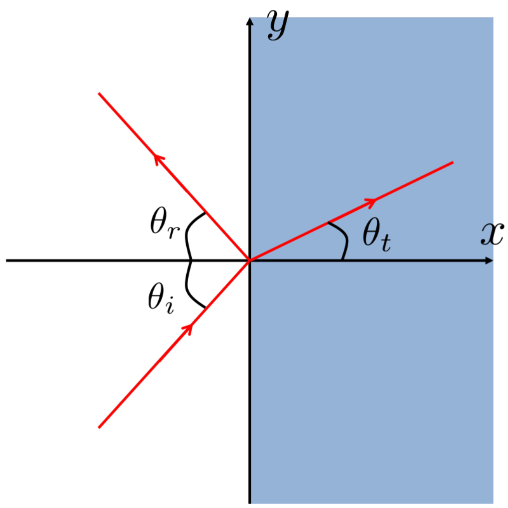
When z_1 and z_2 are real, (1) and (3) add up to -1 while (2) and (4) add up to e^{i2k_2L}. Therefore, whenever $e^{i2k_2L} = 1$, i.e. $L = \frac{n\pi}{k_2}$, A_t^3 and A_r^1 are out of phase and interference destructively.

Question: is it possible to have no transmission? Answer: not with finite $z_1 = z_3$ and z_2.

Question: when $z_1 \neq z_3$, can we choose z_2 such that there is no reflection? Answer: yes! If we choose $z_2 = \sqrt{z_1 z_3}$ and $e^{i2k_2L} = -1$, $A_r = 0$. (Non-reflective coating.)

6.4 Non-perpendicular incident wave, refraction

Demo 20704: Refraction, Transmission, and Reflection with Tabletop Laser



Now consider the more complicated case of transverse wave on a two dimensional membrane. Suppose that we have two such membranes in the xy plane with wave equations

$$\frac{\partial^2\psi}{\partial t^2} = v_1^2\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right), \quad \frac{\partial^2\psi}{\partial t^2} = v_2^2\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) \tag{3}$$

respectively and they are connected along the $x = 0$ axis.

Plane wave of the form

$$\psi(x,t) = Ae^{i(\omega t - \vec{k}\cdot\vec{x})} \tag{4}$$

can exist in both membranes. When the plane wave crosses the boundary at $x = 0$ from one membrane to another, there can be reflection and the transmitted wave may not proceed in the same direction as the incident wave. This is the phenomena of refraction and we are going to find out below the direction of the reflected and transmitted wave.

Consider an incident wave of the form

$$\psi_i(\vec{x},t) = A_i e^{i(\omega t - \vec{k}_i\cdot\vec{x})} \tag{5}$$

where $\omega = v_1|\vec{k}_i|$. Suppose that the reflected wave and transmitted wave take the form

$$\psi_r(\vec{x},t) = A_r e^{i(\omega t - \vec{k}_r\cdot\vec{x})}, \quad \psi_t(\vec{x},t) = A_t e^{i(\omega t - \vec{k}_t\cdot\vec{x})} \tag{6}$$

where $\omega = v_1|\vec{k}_r| = v_2|\vec{k}_t|$.

To satisfy continuity along the whole boundary of $x = 0$, we require that

$$\psi_i(0,y,t) + \psi_r(0,y,t) = \psi_t(0,y,t) \tag{7}$$

That is

$$A_i e^{-ik_i^y y} + A_r e^{-ik_r^y y} = A_t e^{-ik_t^y y} \tag{8}$$

In order for this to be true for any y, we must have

$$k_i^y = k_r^y = k_t^y \tag{9}$$

and at the same time

$$A_i + A_r = A_t \tag{10}$$

From this we can find the direction of the reflected and transmitted wave. Suppose that the wave vector of the incident wave is $\vec{k}_i = (k_i^x, k_i^y)$. It makes an angle θ_i with the normal direction of the interface

$$\tan\theta_i = \frac{k_i^x}{k_i^y} \tag{11}$$

The wave vector of the reflected wave has the same magnitude ω/v_1 and the same y direction component. Therefore, $k_r^x = -k_i^x$.

$$\tan\theta_r = \frac{|k_r^x|}{k_r^y} = \tan\theta_i \tag{12}$$

The wave vector of the transmitted wave has a different magnitude $|\vec{k}_t| = \omega/v_2$. Therefore, θ_t satisfies

$$\sin\theta_t = \frac{k_t^y}{|\vec{k}_t|} = \frac{k_i^y}{|\vec{k}_i|}\frac{v_2}{v_1} = \sin\theta_i\frac{v_2}{v_1} \tag{13}$$

That is

$$\frac{\sin\theta_i}{v_1} = \frac{\sin\theta_t}{v_2} \tag{14}$$

If $v_2 < v_1$, for a given θ_i, we can always find a θ_t that satisfies this condition. However, if $v_2 > v_1$, this may not be the case. In particular if $\frac{v_2}{v_1}\sin\theta_i > 1$, or equivalently

$$\theta_i > \theta_c = \arcsin\frac{v_1}{v_2} \tag{15}$$

then θ_t does not exist. What happens in this case is that the incident wave is completely reflected, with no transmitted wave at all. This is the phenomena of total internal reflection. Total internal reflection is very useful for keeping a wave propagating along a waveguide without leaking out of the wave guide.

Demo: dispersion

7 Interference and Diffraction

In our discussion of multiple reflection, we have already seen a situation where the total (reflected) wave is the sum of many components. The different components can have different phase shifts between them, resulting in different amplitude for the total wave. If all the components are in phase, then they interfere constructively, giving rise to a large total wave; if the components are out of phase with each other, they interfere destructively, giving rise to a small total wave. In general, this kind of phenomena is called interference. In this section, we are going to discuss how interference can lead to different interference and diffraction patterns.

7.1 Cylindrical and spherical wave

Recall that in 1D, to generate a wave that travels in the $+x$ direction, we can drive the $x = 0$ point to oscillate as $Ae^{i\omega t}$. The wave generated is then $Ae^{i(\omega t - kx)}$. Similarly, in 2D and 3D, to generate a plane wave that travels in the $+x$ direction, we can drive the $x = 0$ line / plane to oscillate as $Ae^{i\omega t}$ and the wave generated would be $Ae^{i(\omega t - kx)}$, where $k = \omega/v$. More generally, we can drive the $x = 0$ line / plane as $Ae^{i(\omega t - k_y y)}$ / $Ae^{i(\omega t - k_y y - k_z z)}$. The wave generated is still a plane wave, taking the form $Ae^{i(\omega t - k_x x - k_y y)}$ / $Ae^{i(\omega t - k_x x - k_y y - k_z z)}$, where $k_x = \sqrt{k^2 - k_y^2}$ / $k_x = \sqrt{k^2 - k_y^2 - k_z^2}$, $k = \omega/v$. At a particular t, if we collect all the points with the same phase, we get parallel planes with distance $\lambda = \frac{2\pi}{k}$ between them. Such planes are called wavefronts in the plane wave.

What if we drive only one point $x = 0, y = 0, (z = 0)$ as $Ae^{i\omega t}$? What kind of wave is generated?

In 2D, if we drive $x = 0, y = 0$ as $Ae^{i\omega t}$, a cylindrical wave is generated which takes the form

$$\psi(\vec{x}, t) = \frac{A}{\sqrt{r}} e^{i(\omega t - kr)} \tag{16}$$

where $r = \sqrt{x^2 + y^2}$. The wavefront are circles with distance $\lambda = \frac{2\pi}{k}$ between them.

In 3D, if we drive $x = 0, y = 0, z = 0$ as $Ae^{i\omega t}$, a spherical wave is generated which takes the form

$$\psi(\vec{x}, t) = \frac{A}{r} e^{i(\omega t - kr)} \tag{17}$$

where $r = \sqrt{x^2 + y^2 + z^2}$. The wavefronts are spheres with distance λ between them.

The $\frac{1}{\sqrt{r}}$ and $\frac{1}{r}$ scaling of the amplitude is to satisfy energy conservation: the energy flux (energy flow per unit time and unit area) is proportional to amplitude squared in a wave. The surface area across which energy flows scales as r in 2D and r^2 in 3D. Therefore, to ensure energy conservation, the amplitude scales as $\frac{1}{\sqrt{r}}$ in 2D and $\frac{1}{r}$ in 3D.