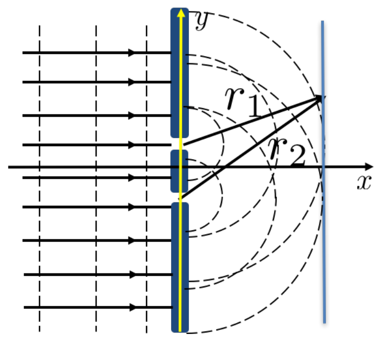
7 Interference and Diffraction

7.2 Double slits

Now consider two small slits on the plane $x = 0$, with a distance a between them. If we shine a plane wave in the x direction, what would we observe on a plane at $x = L$?



check this calculation

Each slit makes a cylindrical wave. At a point (L, y), the total wave is a superposition of the two

$$\frac{A}{\sqrt{r_1}}e^{i(\omega t - kr_1)} + \frac{A}{\sqrt{r_r}}e^{i(\omega t - kr_2)} \tag{1}$$

We can ignore the difference in amplitude, but the difference in phase is important. Assuming that $\lambda << a, y << L$, we can approximate r_1 as

$$r_1 = \sqrt{L^2 + (y - a/2)^2} = \sqrt{L^2 + y^2 - ya + a^2/4} \approx r\left(1 - \frac{1}{2}\frac{ya}{r^2} + \frac{1}{8}\frac{a^2}{r^2}\right) \approx r - \frac{y}{L}\frac{a}{2} + \frac{1}{8}\frac{a^2}{L} \tag{2}$$

Similarly, we find

$$r_2 = r + \frac{y}{L}\frac{a}{2} + \frac{1}{8}\frac{a^2}{L} \tag{3}$$

Then the total wave becomes (the overall phase shift of $k\left(\frac{1}{8}\frac{a^2}{L}\right)$ is ignored)

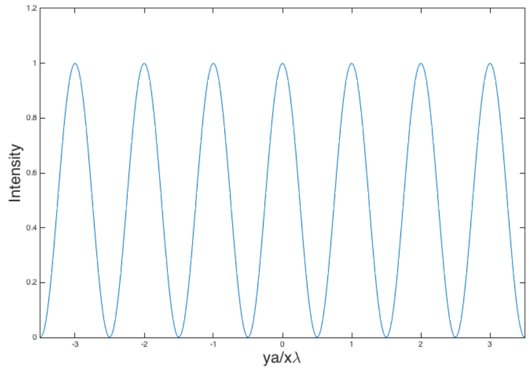
$$\frac{A}{\sqrt{r}}e^{i(\omega t - kr)}\left(e^{ik\frac{y}{L}\frac{a}{2}} + e^{-ik\frac{y}{L}\frac{a}{2}}\right) = \frac{2A}{\sqrt{r}}e^{i(\omega t - kr)}\cos\left(k\frac{y}{L}\frac{a}{2}\right) \tag{4}$$

The time averaged intensity is then proportional to

$$I \propto \cos^2\left(\frac{ka}{2L}y\right) \tag{5}$$

In this interference pattern, the intensity reaches its peak at $y = \frac{n\lambda x}{a}$. This is easy to understand: at the peaks, the optical path difference of the two waves are

$$a\frac{y}{L} = n\lambda \tag{6}$$



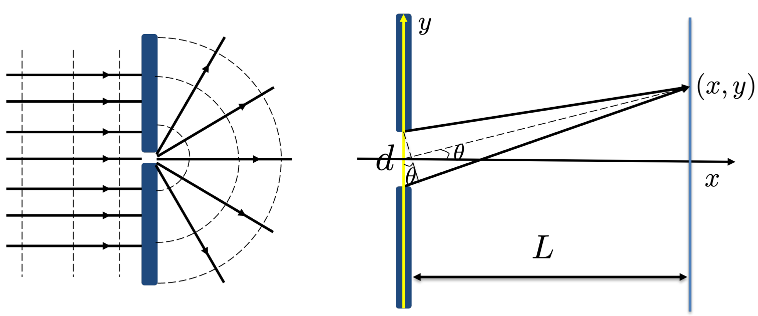
resulting in a phase difference of $n2\pi$. The two waves interference constructively and we see a peak in the intensity.

Half way between the peaks, where $y = \frac{(2n+1)\lambda x}{2a}$, the two waves have an optical path difference of half wave length, resulting in a π phase difference. The two wave interference destructively and we see a zero in the intensity.

All peaks are regularly spaced, with the same width and the same height.

Demo: Ripple Tank / Water drum interference pattern, Michaelson interferometer

7.3 Single slit diffraction



Consider a wall with a slit. Suppose we send a plane wave in the $+x$ direction. It hits the wall at $x = 0$. If the slit is infinitely small, it forms a point source (in 2D) for wave in the region $x > 0$. The wave generated is, at $x > 0$,

$$\psi(x, y, t) = \frac{A}{\sqrt{r}} e^{i(\omega t - kr)} \tag{7}$$

The intensity is uniform in all directions!

This is the phenomena of diffraction. Light after passing through the slit does not travel purely in the $+x$ direction any more. This is a strong manifestation of the wave nature of light.

In reality , we do expect that the angular spread of the wave after passing through the slit to be finite. This is because every slit has a finite width d. What is the intensity distribution pattern for diffraction at a finite width slit?

Imagine dividing the slit into N pieces. As $N \to \infty$, each piece generates a cylindrical wave. The total wave is then a superposition of waves from each piece of the slit.

$$\psi(L, y, t) = \int_{-\frac{d}{2}}^{\frac{d}{2}} A d\tilde{y} \frac{1}{\sqrt{r_{\tilde{y}}}} e^{i(\omega t - k r_{\tilde{y}})} \tag{8}$$

Here $(0, \tilde{y})$ is the location of the piece in the slit, (L, y) is where we are measuring the wave, $r_{\tilde{y}} = \sqrt{L^2 + (y - \tilde{y})^2}$.

This is a complicated integral, but let's consider the case of $L >> y >> \tilde{y}$. Correspondingly, $r = \sqrt{L^2 + y^2} >> y >> \tilde{y}$,

$$r_{\tilde{y}} \approx \sqrt{L^2 + y^2 - 2y\tilde{y}} = r\sqrt{1 - \frac{2y\tilde{y}}{r^2}} \approx r(1 - \frac{y\tilde{y}}{r^2}) \approx r - \tilde{y}\frac{y}{L} \tag{9}$$

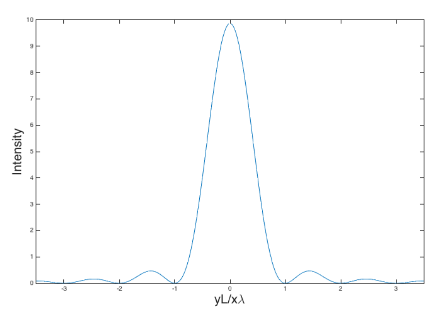
The variation in amplitude due to the factor $\frac{1}{\sqrt{r_{\tilde{y}}}}$ is not important, we can take it simply to be a uniform factor $\frac{1}{\sqrt{r}}$. The difference in phase due to $kr_{\tilde{y}}$, on the other hand, is very important and cannot be ignored.

With these simplifications, the integral reduces to

$$\begin{aligned} \psi(L, y, t) &= \int_{-\frac{d}{2}}^{\frac{d}{2}} A d\tilde{y} \frac{1}{\sqrt{r}} e^{i(\omega t - kr)} e^{ik\tilde{y}y/L} \\ &= \frac{A}{\sqrt{r}(iky/L)} e^{i(\omega t - kr)} e^{ik\tilde{y}y/L} \Big|_{-\frac{d}{2}}^{\frac{d}{2}} \\ &= \frac{2AL}{\sqrt{r}k} e^{i(\omega t - kr)} \frac{\sin\left(\frac{kd}{2L} y\right)}{y} \end{aligned} \tag{10}$$

The time averaged intensity is then proportional to

$$I \propto \frac{\sin^2\left(\frac{kd}{2L} y\right)}{y^2} \tag{11}$$



From this formula (and the plot), we see that $y = 0$ is an intensity maximum, $I_{max} \propto \frac{k^2 d^2}{4L^2}$. The intensity becomes zero at regularly spaced points

$$\frac{kd}{2L} y = n\pi, \ (n \neq 0) \Rightarrow \ y = \frac{2n\pi L}{kd} = n\lambda\frac{L}{d}, \ (n \neq 0) \tag{12}$$

This is the so-called diffraction pattern of a single slit. The central peak is twice as wide as that of the side peaks. At large y, the envelope of the diffraction patterns decays very quickly as $\frac{1}{y^2}$. As $d \to 0$, the central peak becomes infinitely wide and we recover the point source limit.

In this analysis of single slit diffraction pattern, we have used the Huygens principle, which states that every point on a wave front (the slit) is itself the source of cylindrical / spherical waves. The total wave is then the superposition of all the component waves. We are going to use this principle to analyze more complicated situations below.

Demo: Diffraction / Fresnel Diffraction and The Arago Spot

7.4 Finite width double slits

What if each of the double slits has finite width L? For a slit with width L, the wave that emanates from each slit is not a simple cylindrical wave. Instead, it makes the diffraction pattern of the form

$$\psi(L, y, t) = \frac{2AL}{\sqrt{r}k} e^{i(\omega t - kr)} \frac{\sin\left(\frac{kd}{2L} y\right)}{y} \tag{13}$$

When we have two slits, centered at $\pm\frac{a}{2}$ respectively, they interfere as

$$\psi_+(L, y, t) + \psi_-(L, y, t) = \frac{2AL}{\sqrt{r_+}k} e^{i(\omega t - kr_+)} \frac{\sin\left(\frac{kd}{2L}(y - a/2)\right)}{y - a/2} + \frac{2AL}{\sqrt{r_-}k} e^{i(\omega t - kr_-)} \frac{\sin\left(\frac{kd}{2L}(y + a/2)\right)}{y + a/2} \tag{14}$$

When $L >> d, a$, we can approximate r_+ as $r - \frac{a}{2}\frac{y}{L}$ and r_- as $r + \frac{a}{2}\frac{y}{L}$. Similar to the previous case, we ignore the difference in amplitude of the two waves so that

$$\frac{2AL}{\sqrt{r_+}k} \frac{\sin\left(\frac{kd}{2L}(y - a/2)\right)}{y - a/2} \approx \frac{2AL}{\sqrt{r_-}k} \frac{\sin\left(\frac{kd}{2L}(y + a/2)\right)}{y + a/2} \approx \frac{2AL}{\sqrt{r}k} \frac{\sin\left(\frac{kd}{2L}(y)\right)}{y} \tag{15}$$

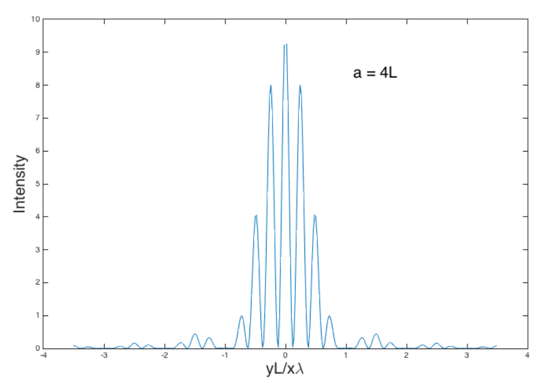
But we need to take into account the difference in phase factor so that

$$\psi_+(L, y, t) + \psi_-(L, y, t) \approx \frac{2AL}{\sqrt{r}k} \frac{\sin\left(\frac{kd}{2L}(y)\right)}{y} e^{i(\omega t - kr)} \cos\left(\frac{ka}{2L} y\right) \tag{16}$$

The total intensity is then proportional to

$$I \sim \frac{\sin^2\left(\frac{kd}{2L}(y)\right)}{y^2} \cos^2\left(\frac{ka}{2L} y\right) \tag{17}$$

That is, the interference pattern is 'modulated' by the diffraction pattern.



7.5 Many slits

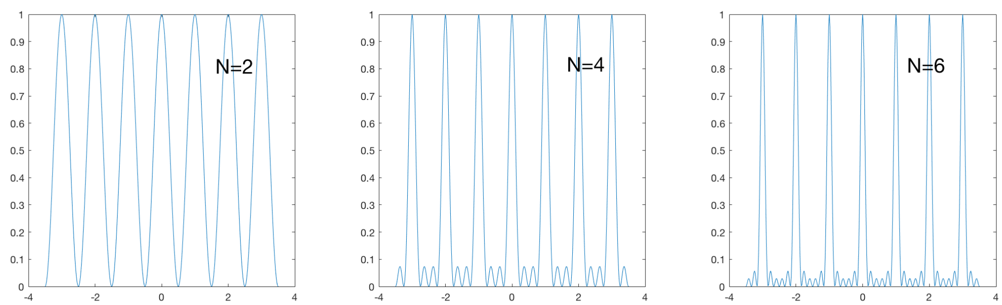
If there are more than 2 slits, what is the interference pattern like?

Consider a set of $2N$ regularly spaced slits. First let's assume that the slits are infinitely narrow and ignore diffraction. When the path difference between different slits is integer multiples of λ, then the waves from different slits interfere constructively with each other, giving rise to a peak in the interference pattern. If we ignore the difference in the amplitude of the waves, then the intensity distribution on the screen is proportional to

$$I \propto \left(e^{ikay/2L} + e^{-ikay/2L} + e^{3ikay/2L} + e^{-3ikay/2L} + ... + e^{(2N-1)ikay/2L} + e^{-(2N-1)ikay/2L}\right)^2 \tag{18}$$

where a is the distance between neighboring slits.

If we make a plot of the intensity vs. $ya/L\lambda$, we see that



the (highest) peak locations are the same, but with more slits, the peaks become sharper.

A useful optical device is made by cutting a large number of very thin slits into an opaque plane. This is called the diffraction grating. An important function of diffraction grating is to separate light of different wavelength. In particular, the location of the second, third ... maximums depend (is proportional to) on the wavelength λ. Therefore, if white light shines perpendicularly on the grating, the central peak is still white, but the second, third etc. peak would be rainbow like: different wave lengths have peak at different locations.

Now if we take into consideration the fact that the slits have finite width, the interference pattern shown above is going to be modulated by the diffraction pattern of each individual slit.